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Circles - Lesson 1

Equations of Circles

LI

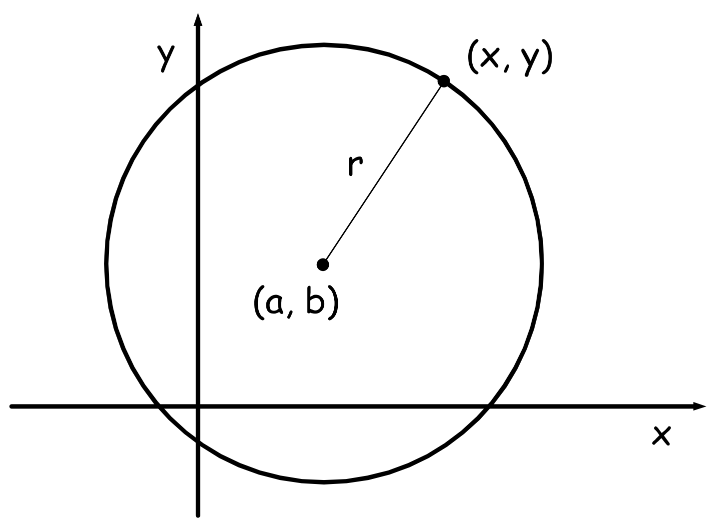
- Know the equation of a circle in its different forms.

SC

- Given the equation of a circle, determine the centre and radius.
- Determine the equation of a circle, given centre and radius.

A circle is the set of points that are the same distance from a fixed point; the fixed point is called the centre of the circle

Consider a circle with centre (a, b), radius of length r and a general point (x, y) on the circle :



Using Pythagoras' Theorem for the triangle made by drawing a vertical line from (x, y) and a horizontal line from (a, b), we have :

$$(x - a)^2 + (y - b)^2 = r^2$$

(Equation of circle with centre (a, b), radius r)

A common situation is when the centre of the circle is at the origin; the equation then becomes :

$$x^2 + y^2 = r^2$$

(Equation of circle with centre $(0, 0)$, radius r)

Note that the radius is always positive, i. e. $r > 0$

Example 1

State the centre and radius of the circle given by :

(a) $(x - 1)^2 + (y - 4)^2 = 49.$

(b) $(x + 3)^2 + y^2 = 121.$

(c) $x^2 + y^2 = 13.$

(a) $(x - 1)^2 + (y - 4)^2 = 49.$

$(x - a)^2 + (y - b)^2 = r^2$

$a = 1, b = 4, r^2 = 49$

$\therefore$ $\boxed{\text{centre} : (1, 4), r = 7}$

(b) $(x + 3)^2 + y^2 = 121.$

$(x - a)^2 + (y - b)^2 = r^2$

$a = -3, b = 0, r^2 = 121$

$\therefore$ $\boxed{\text{centre} : (-3, 0), r = 11}$

(c) $x^2 + y^2 = 13.$

$(x - a)^2 + (y - b)^2 = r^2$

$a = 0, b = 0, r^2 = 13$

$\therefore$ $\boxed{\text{centre} : (0, 0), r = \sqrt{13}}$

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Example 2

Find the equation of the circle with :

(a) centre $(5, 8)$, radius 2.

(b) centre $(-7, 0)$, radius 7.

(c) centre $(0, 0)$, radius $1/2$.

(a) centre $(5, 8)$, radius 2.

$$(x - a)^2 + (y - b)^2 = r^2$$

$$a = 1, b = 4, r = 2$$

$$\therefore \boxed{(x - 5)^2 + (y - 8)^2 = 4}$$

(b) centre $(-7, 0)$, radius 7.

$$(x - a)^2 + (y - b)^2 = r^2$$

$$a = -7, b = 0, r = 7$$

$$\therefore \boxed{(x + 7)^2 + y^2 = 49}$$

(c) centre $(0, 0)$, radius $1/2$.

$$(x - a)^2 + (y - b)^2 = r^2$$

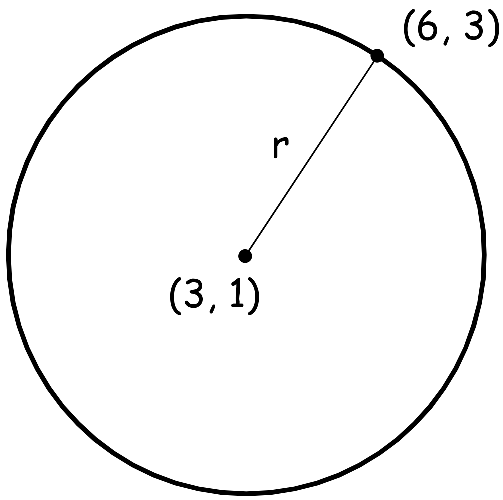
$$a = 0, b = 0, r = 1/2$$

$$\therefore \boxed{\text{centre} : (0, 0), r = 1/4}$$

Example 3

Find the equation of the circle that passes through (6, 3) and has centre (3, 1).

To find the equation of a circle, we need the centre and radius. The centre is given, but the radius needs to be found.



This is done using the Distance Formula :

$$r = \sqrt{(x - a)^2 + (y - b)^2}$$

$$r = \sqrt{(6 - 3)^2 + (3 - 1)^2}$$

$$\underline{r = \sqrt{13}}$$

Hence, the equation of the circle is,

$$\boxed{(x - 3)^2 + (y - 1)^2 = 13}$$

The General Equation of a Circle

Starting with the equation of a circle centre (a, b) and radius r,

$$(x - a)^2 + (y - b)^2 = r^2$$

and expanding the brackets and simplifying, gives,

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 = r^2$$

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

Relabelling $g = -a$, $f = -b$ and $c = a^2 + b^2 - r^2$, we obtain :

$$\boxed{x^2 + y^2 + 2gx + 2fy + c = 0}$$

(General equation of a circle
centre $(-g, -f)$, radius $r = \sqrt{g^2 + f^2 - c}$)

Note that, as $r > 0$, we must have :

$$\boxed{g^2 + f^2 - c > 0}$$

Example 4

Determine whether or not the equation $x^2 + y^2 - 2x + 4y + 9 = 0$ represents a circle.

We just need to calculate the quantity $g^2 + f^2 - c$:

$$x^2 + y^2 - 2x + 4y + 9 = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$2g = -2 \Rightarrow g = -1$$

$$2f = 4 \Rightarrow f = 2$$

$$c = 9$$

$$g^2 + f^2 - c = (-1)^2 + 2^2 - 9$$

$$= 1 + 4 - 9$$

$$= -4$$

As $g^2 + f^2 - c < 0$, the equation $x^2 + y^2 - 2x + 4y + 9 = 0$ does not represent a circle.

Example 5

Find the centre and radius of the circle
$x^2 + y^2 - 4x + 2y + 1 = 0$.

$$x^2 + y^2 - 4x + 2y + 1 = 0$$
$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$2g = -4 \Rightarrow g = -2$$
$$2f = 2 \Rightarrow f = 1$$
$$c = 1$$

Centre : $(-g, -f) = \underline{(2, -1)}$

Radius : $\sqrt{g^2 + f^2 - c}$

$$= \sqrt{(-2)^2 + 1^2 - 1}$$

$$= \underline{2}$$

$$\boxed{\begin{array}{l}\text{Centre : } (2, -1) \\ \text{Radius : } 2\end{array}}$$

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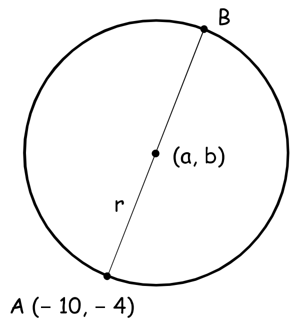
Example 6

The circle $x^2 + y^2 + 4x - 2y - 84 = 0$ has diameter AB. If A is the point $(-10, -4)$, find the coordinates of B.

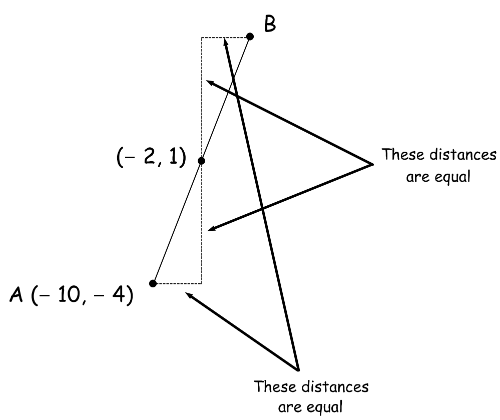


The points A and B are 'symmetrically equidistant' about the centre of the circle. So, find the centre and radius and just 'count coordinates' to get B.

$$x^2 + y^2 + 4x - 2y - 84 = 0$$
$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$2g = 4 \Rightarrow g = 2$$
$$2f = -2 \Rightarrow f = -1$$

Centre : $(-g, -f) = (-2, 1)$



Start at the centre $(-2, 1)$. The x - coordinate of A is 8 units to the left of -2. Therefore the x - coordinate of B is 8 units to the right of -2, i.e. it is equal to $-2 + 8 = 6$. Similarly, as the y - coordinate of A is 5 units down from 1, the y - coordinate of B is 5 units up from 1, i.e. it is equal to $1 + 5 = 6$.

$\boxed{B\ (6, 6)}$

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Example 7

The point $(k, 1)$ lies on the circle $x^2 + y^2 + 6x - 3y = 14$. Find the possible values of k.

$$x^2 + y^2 + 6x - 3y = 14$$

$$k^2 + 1^2 + 6k - 3(1) = 14$$

$$k^2 + 6k - 2 - 14 = 0$$

$$k^2 + 6k - 16 = 0$$

$$(k + 8)(k - 2) = 0$$

$$\boxed{k = -8,\ k = 2}$$

CfE Higher Maths

- pg. 308 - 10 Ex. 14A All Q
- pg. 312 - 3 Ex. 14B All Q