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Further Proof Techniques - Lesson 1

Direct Proof

LI

- Use the method of Direct Proof to prove statements.

SC

- Logical reasoning.

An example (aka instance) is a specific case that satisfies a general statement

WARNING : A general statement is never proven by verifying a specific case.

A direct proof is a way of proving that a certain conclusion follows directly from a set of assumptions using a logically convincing chain of reasoning

Notation

$\mathbb{N}$ - set of all natural numbers.

$\mathbb{W}$ - set of all whole numbers.

$\mathbb{Z}$ - set of all integers.

$\mathbb{Q}$ - set of all rational numbers.

$\mathbb{R}$ - set of all real numbers (all rational and irrational numbers).

$\mathbb{C}$ - set of all complex numbers.

Example 1

Prove that the square of any even number is even.

Let n be any even number. Then $\exists\, k \in \mathbb{Z}$ such that,

$$n = 2k$$

$$\therefore \quad n^2 = (2k)^2$$

$$\Rightarrow \quad n^2 = 4k^2$$

$$\Rightarrow \quad \underline{n^2 = 2(2k^2)}$$

$\boxed{\text{As } k \in \mathbb{Z},\ 2k^2 \in \mathbb{Z};\ \text{hence, } n \text{ even} \Rightarrow n^2 \text{ even}}$

Example 2

Prove that the sum of any two rational numbers is rational.

Let the two rational numbers be x and y. As $x, y \in \mathbb{Q}$,

$\exists\, a, b, c, d \in \mathbb{Z}$ $(b, d \neq 0)$ such that,

$$x = \frac{a}{b} \quad \text{and} \quad y = \frac{c}{d}$$

$$\therefore \qquad x + y = \frac{a}{b} + \frac{c}{d}$$

$$\Rightarrow \qquad \underline{x + y = \frac{ad + bc}{bd}}$$

As $a, b, c, d \in \mathbb{Z}$ $(b, d \neq 0)$, $ad + bc, bd \in \mathbb{Z}$ $(bd \neq 0)$;

hence, $x, y \in \mathbb{Q} \Rightarrow x + y \in \mathbb{Q}$

Example 3

Prove that the square of any odd number plus the cube of any even number is odd.

Let n be any odd number and let m be any even number.
Then $\exists\, k, p \in \mathbb{Z}$ such that,

$$n = 2k + 1 \quad \text{and} \quad m = 2p$$

$$\therefore \quad n^2 + m^3 = (2k + 1)^2 + (2p)^3$$

$$\Rightarrow \quad n^2 + m^3 = 4k^2 + 4k + 1 + 8p^3$$

$$\Rightarrow \quad \underline{n^2 + m^3 = 2(2k^2 + 2k + 4p^3) + 1}$$

As $k, p \in \mathbb{Z}$, $2k^2 + 2k + 4p^3 \in \mathbb{Z}$; hence, n odd and m even $\Rightarrow n^2 + m^3$ is odd

Example 4

Prove that the product of any odd function and any even function is odd.

Let f be any odd function and let g be any even function. Then,

$$f(-x) = -f(x) \quad (\forall x \in \text{dom } f)$$

and

$$g(-x) = g(x) \quad (\forall x \in \text{dom } g)$$

Let $h(x) = f(x)\,g(x)$. We must show that h is an odd function, i.e., that $h(-x) = -h(x)$ $(\forall x \in \text{dom } h)$. Let $x \in \text{dom } h$. Then,

$$h(-x) = f(-x)\,g(-x)$$

$$\Rightarrow \quad h(-x) = (-f(x))\,g(x)$$

$$\Rightarrow \quad h(-x) = -f(x)\,g(x)$$

$$\Rightarrow \quad \underline{h(-x) = -h(x)}$$

As $h(-x) = -h(x)$ $(\forall x \in \text{dom } h)$, the product of an odd and an even function is odd

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Questions

Prove the following using direct proof :

1) The product of any two rational numbers is rational.
2) The difference of any two rational numbers is rational.
3) The quotient of any two non-zero rational numbers is rational.
4) The square of any odd number is odd.
5) The cube of any even number is even.
6) The sum of the squares of any two odd numbers is even.
7) The composition of any two linear functions is linear.
8) The product of any two exponential functions is exponential.
9) The derivative of any cubic function is quadratic.
10) The integral of any quadratic function is cubic.
11) The sum of any two even functions is even.
12) The product of any two odd functions is even.
13) The conjugate of the sum of any two complex numbers equals the sum of the conjugates of those two complex numbers.