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Unit 2 : The Binomial Theorem - Lesson 1

Binomial Coefficients

LI

- Know what a Binomial Coefficient is.
- Use binomial coefficients to solve problems.

SC

- How to count in efficient ways.
- Factorials.

Factorials

Given a whole number n, the factorial of n (aka n factorial) is the natural number,

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

(The convention $0! = 1$ is made)

The factorial is useful in counting objects; more specifically, it represents the number of ways n distinct objects can be arranged (where the order matters).

This explains why $0! = 1$; there is only one way to arrange 0 objects (!).

Example 1

Find the number of ways five cyclists can finish a race (all different times).

There are 5 ways of picking first place; for each of these, there are 4 ways of picking second place; etc. . Hence, the number of required ways is,

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$

$$\Rightarrow \quad \boxed{5! = 120}$$

Permutations

Given whole numbers r and n, the permutation of r objects from n (so $r \leq n$) is,

$$^{n}P_{r} = \frac{n!}{(n - r)!}$$

The permutation of r objects from n distinct ones is the number of ways of choosing r objects from n (where the order matters).

Note that if $r = n$, then $^{n}P_{r} = n!$; the number of ways of choosing n objects from n distinct ones (where order matters).

Example 2

Find the number of ways first and second prizes can be allocated in a race with six cyclists.

Obviously there are 6 distinct objects and 2 of them are chosen (obviously the order matters too). Hence, the required number is,

$$^{6}P_{2} = \frac{6!}{(6 - 2)!}$$

$$\Rightarrow \quad ^{6}P_{2} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}$$

$$\Rightarrow \quad ^{6}P_{2} = 6 \times 5$$

$$\Rightarrow \quad \boxed{^{6}P_{2} = 30}$$

Combinations

Given whole numbers r and n, the combination of r objects from n (so $r \le n$) is,

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

The combination of r objects from n distinct ones is the number of ways of choosing r objects from n (where the order does not matter).

Example 3

A palette of paint contains seven different colours. Find the number of ways four of these colours can be mixed.

Obviously there are 7 distinct objects and for the 4 that are to be mixed, obviously the order in which they are mixed does not matter. Hence, the required number is,

$$^{7}C_{4} = \frac{7!}{4!(7-4)!}$$

$$\Rightarrow \quad ^{7}C_{4} = \frac{7!}{4!3!}$$

$$\Rightarrow \quad ^{7}C_{4} = \frac{7 \times 6 \times 5 \times 4!}{4!.6}$$

$$\Rightarrow \quad ^{7}C_{4} = 7 \times 5$$

$$\Rightarrow \quad \boxed{^{7}C_{4} = 35}$$

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Binomial Coefficients

The natural numbers ${}^{n}C_{r}$ are called Binomial Coefficients (reason for this terminology will be apparent in the next lesson). They are also written as,

$$\boxed{\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}}$$

It is clear from the definition of ${}^{n}C_{r}$ that :

$$\binom{n}{r} = \binom{n}{n-r}$$

$$({}^{n}C_{r} = {}^{n}C_{n-r})$$

It is also true (but not obvious !) that :

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

$$({}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r})$$

Example 4

Solve for n :

$$\binom{n}{2} = 36$$

$$\binom{n}{2} = 36$$

$$\therefore \quad \frac{n!}{2!(n-2)!} = 36$$

$$\Rightarrow \quad \frac{n \times (n-1) \times (n-2)!}{2.(n-2)!} = 36$$

$$\Rightarrow \quad \frac{n(n-1)}{2} = 36$$

$$\Rightarrow \quad n(n-1) = 72$$

$$\Rightarrow \quad n^2 - n = 72$$

$$\Rightarrow \quad n^2 - n - 72 = 0$$

$$\Rightarrow \quad (n-9)(n+8) = 0$$

$$\Rightarrow \quad \underline{n = 9, n = -8}$$

As n is a natural number, n cannot be negative. So,

$$\boxed{n = 9}$$

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Example 5

Show that :

$$^{n}C_{r} + 2\,^{n}C_{r+1} + \,^{n}C_{r+2} = \,^{n+2}C_{r+2}$$

$$\begin{aligned} \text{LHS} &= \,^{n}C_{r} + 2\,^{n}C_{r+1} + \,^{n}C_{r+2} \\ &= \,^{n}C_{r} + \,^{n}C_{r+1} + \,^{n}C_{r+1} + \,^{n}C_{r+2} \end{aligned}$$

Using $^{n}C_{r} + \,^{n}C_{r-1} = \,^{n+1}C_{r}$, the first two terms can be combined (replacing r by $r + 1$); the remaining two terms can be combined by using the same identity (replacing r by $r + 2$). So,

$$\begin{aligned} \text{LHS} &= \,^{n+1}C_{r+1} + \,^{n+1}C_{r+2} \\ &= \,^{n+2}C_{r+2} \\ &= \text{RHS} \end{aligned}$$

$$\therefore \quad \boxed{^{n}C_{r} + 2\,^{n}C_{r+1} + \,^{n}C_{r+2} = \,^{n+2}C_{r+2}}$$

AH Maths - MiA (2nd Edn.)

- pg. 31-2 Ex. 3.2 Q 2, 4 a, 6 a, b, 7.
- pg. 33-4 Ex. 3.3 Q 1, 2, 4, 5, 6, 7 a, b, d.

Ex. 3.2

2 a The combination on the lock on my case uses the four digits 1, 2, 3 and 4. I have forgotten the order in which they appear.
How many different ways can they be arranged?

b In a game of Scrabble a player has seven different letters.
He rearranges them, looking for words.
How many different arrangements can he make of the seven letters?

c A pack of cards has 52 different cards.
Calculate the number of ways these can be arranged, giving your answer correct to 3 significant figures.

4 a From a class of 23 students, three have to be selected to be class representative, secretary and treasurer of the newly formed student committee.
In how many different ways can this be done?

6 a i In the card game *Brag* players are given three cards. Assuming the pack is made up of 52 different cards, how many different hands are possible?

ii In the game *Bridge* players are given 13 cards.
How many different hands are possible?
Give your answer to 3 significant figures.

iii In the game *Solitaire* players are given 52 cards.
How many different hands are possible?

b In the national lottery there are 49 numbers from which to pick a set of six numbers.
How many different sets of six numbers can be picked?

7 A manufacturer of soup has six basic ingredients from which he can pick and mix to make varieties of soup.
He has to use at least two ingredients before he can call it a soup.
How many varieties of soup can he make?

Ex. 3.3

1 Given that $^{10}C_2 = 45$, $^{10}C_7 = 120$ and $^{10}C_4 = 210$ write the values of

a $^{10}C_6$ b $^{10}C_8$ c $^{10}C_3$

2 a From a team of 11 cricketers, two are selected to bat first. How many different ways can this be done?

b From a team of 11 cricketers, nine are selected not to bat first. How many different ways can this be done?

c Complete the statement $^{11}C_2 = {}^{11}C$.

4 Solve these equations.

a $\binom{n}{2} = 6$ b $\binom{n}{2} = 45$ c $\binom{n}{2} = 28$ d $\binom{n}{2} = 120$

e $\binom{2n}{2} = 15$ f $\binom{2n}{2} = 45$ g $\binom{2n}{2} = 66$ h $\binom{2n}{2} = 276$

5 Find a value of n which satisfies these equations.

a $\binom{n}{3} = 4$ b $\binom{n}{3} = 10$ c $\binom{n}{3} = 35$ d $\binom{n}{3} = 120$

6 Make use of the fact that $\binom{n}{n-r} = \binom{n}{r}$ to help you solve these equations.

a $\binom{n}{n-2} = 15$ b $\binom{n}{n-2} = 55$ c $\binom{n}{n-3} = 84$

7 Using the identity $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ find a value of n which satisfies each of these equations.

a $\binom{n}{1} + \binom{n}{2} = 28$ b $\binom{n+1}{1} + \binom{n+1}{2} = 66$

d $\binom{n+1}{2} - \binom{n}{1} = 36$

Answers to AH Maths (MiA), pg. 31-2, Ex. 3.2

2 a 24 b 5040 c $8{\cdot}07 \times 10^{67}$

4 a 10626

6 a i $^{52}C_3 = \dfrac{52!}{3!49!} = 22100$

ii $^{52}C_{13} = \dfrac{52!}{13!39!} = 6{\cdot}35 \times 10^{11}$

iii $^{52}C_{52} = \dfrac{52!}{52!0!} = 1$

b 13 983 816

7 57

Answers to AH Maths (MiA), pg. 33-4, Ex. 3.3

1 a 210 b 45 c 120

2 a 55 b 55 c $^{11}C_2 = {}^{11}C_9$

4 a 4 b 10 c 8 d 16

e 3 f 5 g 6 h 12

5 a 4 b 5 c 7 d 10

6 a 6 b 11 c 9

7 a 7 b 10 d 9