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Unit 2 : Arithmetic and Algebra of Complex Numbers - Lesson 1

Arithmetic of Complex Numbers

LI

- Know what a complex number is.
- Add, subtract, multiply and divide complex numbers.

SC

- Arithmetic of real numbers.

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A complex number z is a number of the form,

$$z = x + y\,i \quad (\text{where } i^2 = -1)$$

where x and y are real numbers and i is the positive root of the equation $z^2 + 1 = 0$ (the variable z is used in complex numbers). The set of all complex numbers is denoted by $\mathbb{C}$ (just as the set of all real numbers is denoted by $\mathbb{R}$).

The real part of z is x : $\mathrm{Re}\,(z) = x$; the imaginary part of z is y : $\mathrm{Im}\,(z) = y$

The complex conjugate of $z = x + y\,i$ is the complex number (denoted by $\bar{z}$) :

$$\bar{z} = x - y\,i$$

When a complex number z is written in the form $x + y\,i$, this is called the Cartesian Form (of z)

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Powers of i

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$i^5 = i$$

etc.

Two complex numbers are equal if they have the same real parts and the same imaginary parts

Addition and Subtraction of Complex Numbers

Complex numbers are added or subtracted by adding or subtracting their real parts and imaginary parts separately :

$$\boxed{z = a + b\,i,\ w = c + d\,i \Rightarrow z \pm w = (a \pm c) + (b \pm d)\,i}$$

Multiplication of Complex Numbers

Complex numbers are multiplied by expanding brackets and collecting together real and imaginary parts :

$$\boxed{z = a + b\,i,\ w = c + d\,i \Rightarrow z\,w = (ac - bd) + (ad + bc)\,i}$$

This formula is not to be memorised; just expand brackets and simplify.

Division of Complex Numbers

To divide two complex numbers, multiply the numerator and denominator by the complex conjugate of the denominator.

Example 1

If $z = 3 + 2i$ and $w = 5 - i$, find :

(a) $z + w$.

(b) $3z - 5w$.

(a)

$$z + w = 3 + 2i + 5 - i$$

$$\Rightarrow \quad \boxed{z + w = 8 + i}$$

(b)

$$3z - 5w = 3(3 + 2i) - 5(5 - i)$$

$$\Rightarrow \quad 3z - 5w = 9 + 6i - 25 + 5i$$

$$\Rightarrow \quad \boxed{3z - 5w = -16 + 11i}$$

Example 2

If $z = -2 + 4i$ and $w = 1 + i$, find :

(a) zw.

(b) $z \div w$.

(a)

$$zw = (-2 + 4i)(1 + i)$$

$$\Rightarrow \quad zw = -2 - 2i + 4i + 4i^2$$

$$\Rightarrow \quad zw = -2 - 2i + 4i - 4$$

$$\Rightarrow \quad \boxed{zw = -6 + 2i}$$

(b)

$$z \div w = \frac{-2 + 4i}{1 + i}$$

$$\therefore \quad z \div w = \frac{(-2 + 4i)(1 - i)}{(1 + i)(1 - i)}$$

$$\Rightarrow \quad z \div w = \frac{-2 + 2i + 4i + 4}{1 + 1}$$

$$\Rightarrow \quad z \div w = \frac{2 + 6i}{2}$$

$$\Rightarrow \quad \boxed{z \div w = 1 + 3i}$$

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Example 3

If $a + bi = (4 + i)^2$, find a and b.

$$a + bi = (4 + i)^2$$

$$\Rightarrow \quad a + bi = (4 + i)(4 + i)$$

$$\Rightarrow \quad a + bi = 16 + 4i + 4i - 1$$

$$\Rightarrow \quad \underline{a + bi = 15 + 8i}$$

Equating real and imaginary parts gives,

$$\boxed{a = 15,\ b = 8}$$

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Example 4

Show that for complex numbers z and w,

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$z = a + b\,i,\ w = c + d\,i \Rightarrow \bar{z} = a - b\,i,\ \bar{w} = c - d\,i$$

$$\overline{z + w} = \overline{(a + b\,i) + (c + d\,i)}$$

$$\Rightarrow \quad \overline{z + w} = \overline{(a + c) + (b + d)\,i}$$

$$\Rightarrow \quad \overline{z + w} = (a + c) - (b + d)\,i$$

$$\Rightarrow \quad \overline{z + w} = (a - b\,i) + (c - d\,i)$$

$$\Rightarrow \quad \boxed{\overline{z + w} = \bar{z} + \bar{w}}$$

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AH Maths - MiA (2^{nd} Edn.)

- pg. 207-8 Ex. 12.1 Q 1, 2, 6, 7, 8.
- pg. 209 Ex. 12.2 Q 1 - 3.

Ex. 12.1

1 Given $z_1 = 2 + i$ and $z_2 = 3 + 4i$ calculate these in the form $a + bi$.

a $z_1 + z_2$ b $z_1 z_2$ c $3z_1$ d $2z_2$ e $4z_1 + 3z_2$ f ${z_1}^2$

g ${z_1}^3$ h ${z_1}^3 z_2$ i $-z_2$ j $z_1 - z_2$ k $z_2 - z_1$ l ${z_1}^2 - 2z_2$

2 Simplify these, expressing your answer in the form $a + bi$.

a $(3 + 4i) + (1 + i)$ b $(6 - 2i) + (4 + 2i)$ c $(1 + i)(1 - i)$

d $(1 + 2i)(1 - 3i)$ e $(4 - 3i)^2$ f $2(3 - i) - 4(1 + 2i)$

g $3(1+ i) - i(1 + 3i)$ h $2i(2 + 3i)(1 - 2i)$ i $(3 - i)^2(3 + i)$

6 a Simplify each of these.

i $(3 + i)(3 - i)$ ii $(2 + 3i)(2 - 3i)$ iii $(1 + 2i)(1 - 2i)$

b Comment on your answers in each case. Make a conjecture.

c Simplify $(a + ib)(a - ib)$ to prove your conjecture.

7 a $i = i$; $i^2 = -1$; $i^3 = i \times i^2 = -i$; $i^4 = i^2 \times i^2 = 1$

Work out the powers of i up to i^{12}.

b Given that n is an integer, evaluate

i i^{4n-1} ii i^{4n+1} iii i^{4n+2} iv i^{4n} v i^{4n+3}

8 By equating real and imaginary parts, find a and b in each case.

a $a + bi = (3 + i)^2$ b $a + bi = (3 + 2i)^2$ c $a + bi = (2 + i)(3 + 4i)$

Ex. 12.2

1 Calculate these divisions, expressing your answer in the form $a + ib$ where $a, b \in \mathbb{R}$.

a $(8 + 4i) \div (1 + 3i)$ b $(8 + i) \div (3 + 2i)$ c $(6 + 2i) \div (4 - 2i)$

d $(-1 - 3i) \div (1 - 2i)$ e $8 \div (1 + 2i)$ f $(6 + i) \div (3 - i)$

2 In each case below, express z^{-1} in the form $a + ib$ where $a, b \in \mathbb{R}$.

a $z = i$ b $z = 1 - i$ c $z = 2 + 2i$ d $z = 3 + i$ e $z = 4 - 2i$

3 Simplify

a $\dfrac{17 - 7i}{5 + i}$ b $\dfrac{21 + 9i}{2 + 5i}$ c $\dfrac{7 - 3i}{1 + i}$

d $\dfrac{2 - 5i}{1 + i}$ e $\dfrac{3 - 2i}{1 + 2i}$ f $\dfrac{3}{3 + 4i}$

Answers to AH Maths (MiA), pg. 207-8, Ex. 12.1

1 **a** $5+5i$ **b** $2+11i$ **c** $6+3i$
d $6+8i$ **e** $17+16i$ **f** $3+4i$
g $2+11i$ **h** $-38+41i$ **i** $-3-4i$
j $-1-3i$ **k** $1+3i$ **l** $-3-4i$

2 **a** $4+5i$ **b** 10 **c** 2
d $7-i$ **e** $7-24i$ **f** $2-10i$
g $6+2i$ **h** $2+16i$ **i** $30-10i$

6 **a** **i** 10 **ii** 13 **iii** 5 **b** All answers are real.
c $a^2+b^2 \in \mathrm{R}$

7 **a** $i^1=i, i^2=-1, i^3=-i, i^4=1, i^5=i, i^6=-1, i^7=-i,$
$i^8=1, i^9=i, i^{10}=-1, i^{11}=-i, i^{12}=1$
b **i** $i^{4n-1}=-i$ **ii** $i^{4n+1}=i$ **iii** $i^{4n+2}=-1$
iv $i^{4n}=1$ **v** $i^{4n+3}=-i$

8 **a** $a=8, b=6$ **b** $a=5, b=12$
c $a=2, b=11$

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Answers to AH Maths (MiA), pg. 209, Ex. 12.2

1	**a**	$2 - 2i$	**b**	$2 - i$	**c**	$1 + i$
	d	$1 - i$	**e**	$1{\cdot}6 - 3{\cdot}2i$	**f**	$1{\cdot}7 + 0{\cdot}9i$
2	**a**	$-i$	**b**	$0{\cdot}5 + 0{\cdot}5i$	**c**	$0{\cdot}25 - 0{\cdot}25i$
	d	$0{\cdot}3 - 0{\cdot}1i$	**e**	$0{\cdot}2 + 0{\cdot}1i$		
3	**a**	$3 - 2i$	**b**	$3 - 3i$	**c**	$2 - 5i$
	d	$-1{\cdot}5 - 3{\cdot}5i$	**e**	$-0{\cdot}2 - 1{\cdot}6i$	**f**	$0{\cdot}36 - 0{\cdot}48i$

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