SECTION I

Attempt ALL questions in this Section.

Marks

1. Differentiate the following functions with respect to x.

   (*a*) $y = x^3 e^{-x^2}$

   (*b*) $f(x) = \tan^{-1}\left(\sqrt{x-1}\right), \quad x>1$

   (*c*) $f(x) = \dfrac{x^2}{\cos x}, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$. **2,2,2**

2. The Woollen Mill's *mad-January Sale* started on Thursday January 2nd. The number of sweaters sold on each of the 30 days of the sale is tabulated below where the sales are recorded to the nearest dozen.

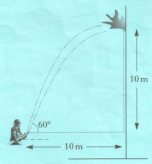
January 1997					
Sunday		41	37	18	21
Monday		25	5	7	5
Tuesday		20	11	8	6
Wednesday		23	17	9	3
Thursday	40	19	10	12	8
Friday	59	17	7	7	12
Saturday	46	39	22	17	

Construct a stem-and-leaf diagram for this data and state the median. **2,1**

3. Use the central difference method with $h = 0{\cdot}1$ to obtain an approximate value for $f'(2)$ where $f(x) = \cos^2(\tan^{-1}x)$, rounding your answer to three decimal places. **4**

Marks

4. Firefighters are tackling a fire in a third-floor chemical laboratory.



The seat of the blaze is 10 metres above the nozzle of the hose and, to avoid falling debris, the firefighters are forced to stand back 10 metres from the wall. If the nozzle of the hose makes an angle of 60° with the horizontal, find the least velocity with which the water must leave the hose if it is to reach the fire.

[You may assume that the water droplets act as projectiles and you may ignore air resistance.

You should take the magnitude of the acceleration due to gravity to be $9{\cdot}8\ \mathrm{m\,s^{-2}}$.] **5**

5. By means of the substitution $u = x^2 - 8$, find

$$\int_3^4 x^3 (x^2 - 8)^{\frac{1}{3}}\, dx.$$ **5**

6. Let A be the matrix $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

   Show that $A^2 - 7A = cI$ where c is a real number and I is the 2×2 identity matrix. **3**

   By considering this equation in the form

$$A(A-7I) = cI,$$

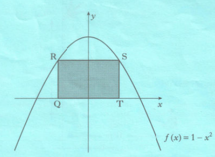
   obtain the matrix B for which

$$AB = I.$$ **2**

[Turn over

Marks

7. The rectangle QRST is situated beneath the graph of the function $f(x) = 1 - x^2$ as shown in the diagram below.



   Let T be the point $(t, 0)$. Find the value of t for which the rectangle has the largest area and find this largest area. **5**

8. Every Friday evening, two friends Paul and Elspeth play each other at tiddlywinks. They do not play for long, in fact the match ends as soon as one player has won two games. For any game the probability that Elspeth wins is $0{\cdot}6$.

   (*a*) Draw a tree diagram showing possible outcomes for one match. **2**

   (*b*) What is the probability that Elspeth wins a particular match? **2**

   (*c*) What is the probability that Paul wins two consecutive matches? **1**

9. (*a*) Find partial fractions for

$$\frac{4}{x^2-4}.$$ **2**

   (*b*) By using (*a*) obtain

$$\int \frac{x^2}{x^2-4}\, dx.$$ **4**

10. For $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + x,\ x \in \mathbf{R}$, find all the values of x for which $|f'(x)| < 1$.

    [A solution to this question which relies on a graphical or numerical approach will receive little or no credit.] **6**

Marks

11. (*a*) Prove that $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$.

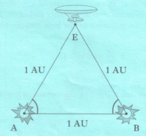
    [You may assume that $\sin(a+b) = \sin a \cos b + \cos a \sin b$ and $\cos(a+b) = \cos a \cos b - \sin a \sin b$.] **2**

    (*b*) Hence, given that

$$\tan(x+45)^\circ = \frac{1}{2},$$

    find the **exact** value of $\tan x^\circ$. **3**

12. A star system consists of a pair of stars, A and B, which are 1 astronomical unit (AU) apart. The mass of A is 3 solar masses and that of B is 5 solar masses. A starship is at a point E which is 1 AU from A and 1 AU from B.



    The force on a starship of mass m due to a star of mass M is of magnitude $\dfrac{mMG}{r^2}$ towards the star, where m, M are measured in solar masses and r is the distance measured in astronomical units. The unit of force has been adjusted so that G, the universal constant of gravitation, may be taken as 1.

    Calculate, in terms of m, the magnitude of the resultant force on the starship due to the pull of the two stars and the angle this force makes with the line AB joining the stars. **5**

[Turn over

SECTION II

Attempt ALL questions in this Section.

Marks

13. (*a*) Find the modulus and argument of the complex number $2 + 2\sqrt{3}i$ and plot it on an Argand diagram. **3**

    (*b*) By writing $z = r(\cos\theta + i \sin\theta)$, obtain a value for r and values for θ such that

$$z^3 = 2 + 2\sqrt{3}i.$$

    Plot the complex numbers z on the same Argand diagram as in part (*a*). **6**

14. An accident at a factory on a river results in the release of a polluting chemical. Immediately after the accident, the concentration of the chemical in the river becomes $k\,\mathrm{g/m^3}$. The river flows at a constant rate of $w\,\mathrm{m^3}$/hour into a loch of volume $V\,\mathrm{m^3}$. Water flows over the dam at the other end of the loch at the same rate. The level, $x\,\mathrm{g/m^3}$, of the pollutant in the loch t hours after the accident satisfies the differential equation

$$V\frac{dx}{dt} = w(k-x).$$

    (*a*) Find the general solution for x in term of t. **4**

    (*b*) In this particular case, the values of the constants are $V = 16\,000\,000$, $w = 8000$ and $k = 1000$.

    (i) Before the accident, the level of chemical in the loch was zero. Fish in the loch will be poisoned if the level of pollutant in the loch reaches $10\,\mathrm{g/m^3}$. How long do the authorities have to stop the leak before this level is reached? **3**

    (ii) In fact, when the level of the pollutant in the loch has reached $5\,\mathrm{g/m^3}$, the leak is located and plugged. The level in the river then drops to zero and the level in the loch falls according to the differential equation

$$V\frac{dx}{dt} = -wx.$$

    According to European Union standards, $1\,\mathrm{g/m^3}$ is a safe level for the chemical. How much longer will it be before the level in the loch drops to this value? **5**

Marks

15. Make use of the fact that

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

    to write down a formula for $\sum_{k=1}^{n} k^3$ in terms of n. **2**

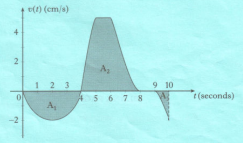
    Use this formula to obtain an expression for $\sum_{k=1}^{n}(2k^3 + k^2 - k)$ in terms of n.

    Express your answer in fully factorised form. **6**

    Hence evaluate

$$(2+1-1) + (16+4-2) + \ldots + (2000+100-10).$$ **2**

16.



    The graph above represents the **velocity** v of the cutting head of a computer controlled milling machine. **Positive** values of v represent movement to the right.

    (*a*) What is represented, in terms of the movement of the cutting head, by the gradient $\dfrac{dv}{dt}$ of this graph at a time t? **1**

    (*b*) What is represented by the integral $\int_0^T v(t)\,dt$ for some time T? **2**

    The areas marked on the graph have values:

$$A_1 = 6, \quad A_2 = 13, \quad A_3 = 1.$$

    (*c*) What is the maximum displacement of the cutting head from its starting position and what is the smallest value of t for which this maximum is attained? **2**

    (*d*) What is the maximum acceleration of the cutting head? When this is attained, is the head moving to the right or to the left? **2**

    (*e*) What is the total distance moved by the cutting head for $0 \le t \le 10$? **2**

[*END OF QUESTION PAPER*]