SECTION I

Attempt ALL questions in this Section.

Marks

1. Determine the modulus and argument of the complex number $\frac{1}{1-i}$. **3**

2. A car is travelling along a straight road. Its acceleration at time t seconds is

$$a(t) = 20 + 6t + 4t^2,$$

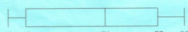
measured in metres per second per second. The car started from rest at time $t = 0$ from a point O on the road. Find the speed of the car and its distance from O after t seconds. **4**

3. (*a*) Draw a horizontal box plot to represent the following ordered data which arise from attendance at a public swimming pool on Monday mornings throughout the first half of 1991.

12	13	15	16	18	18	19	20	21	21
22	23	25	26	26	26	27	27	28	29
30	30	31	32	63	72				

3

(*b*) The box plot given below represents the attendance on Saturday mornings throughout the first half of 1991.



60 62 71 77 80

Compare the box plots in parts (*a*) and (*b*) and suggest **two** reasons for any differences. **2**

Marks

4. By using the substitution $x = \frac{1}{3}\sin\theta$, evaluate

$$\int_0^{1/6} \frac{1}{(1-9x^2)^{\frac{1}{2}}}\,dx.$$ **4**

5. Given that

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 1 \\ 0 & 0 & k \end{pmatrix},$$

calculate $\mathbf{A}^2$ and find the values of k for which the determinant of the matrix $\mathbf{A}^2 - 2\mathbf{I}$ is zero. ($\mathbf{I}$ is the 3×3 identity matrix.) **5**

6. Show that the function

$$y = \frac{\sin(kx)}{x},$$

where $x \neq 0$ and k is a non-zero constant, satisfies the differential equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + k^2y = 0.$$ **5**

[Turn over

Marks

7. A friend asks for your advice concerning the weather during a forthcoming three-day holiday weekend.

By using a tree diagram, or otherwise, calculate:

(*a*) the probability that all three days will be wet; **1**

(*b*) the probability that there will be only one wet day; **2**

(*c*) the probability that at least two days will be dry. **3**

You may assume that the probability of a wet day is 0·3 and that the weather on each day is independent of the weather on the other days.

8. A body is acted upon by two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ where

$$\mathbf{F}_1 = -7\mathbf{i} + 2\mathbf{j} \text{ and } \mathbf{F}_2 = 2\mathbf{i} + 4\mathbf{j}$$

and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions of Ox and Oy respectively.

Determine the magnitude and direction of the resultant force acting on the body. **3**

Illustrate your answer with a sketch. **1**

9. Use the composite trapezium rule with 4 sub-intervals to obtain an approximation to

$$\int_1^2 \frac{e^x}{x}\,dx$$

rounding your answer to two decimal places. **4**

10. Use integration by parts to evaluate

$$\int_0^{\pi} x\sin x\,dx.$$ **4**

11. Given that $z = -2 + i$ is one root of

$$z^4 + z^3 - z^2 + 9z + 30 = 0,$$

find all the remaining roots. **6**

12. Given that $y = e^x\sin(\sqrt{3}x)$, find the values of x in the interval $0 < x < \pi$ such that $\frac{dy}{dx} = 0$. **4**

13. Prove that the equation $x^3 - 2x - 1 = 0$ has only one root in the interval $1 < x < 3$. **2**

Verify that the equation can be rewritten as $x = (2x+1)^{\frac{1}{3}}$. **1**

By using the iterative scheme $x_{n+1} = (2x_n + 1)^{\frac{1}{3}}$ with $x_0 = 2$, obtain an approximation to the root which is correct to two decimal places. **3**

SECTION II

Attempt ALL questions in this Section.

Marks

14. A large population of N individuals contains, at time $t = 0$, just one individual with a contagious disease. Assume that the spread of the disease is governed by the equation

$$\frac{dn}{dt} = k(N-n)n$$

where $n(t)$ is the number of infected individuals after a time t days and k is a constant.

(*a*) Find n explicitly as a function of t. **7**

(*b*) Given that measurements reveal that half the population is infected after 100 days, show that

$$k = \{\ln(N-1)\}/100N.$$ **2**

(*c*) What value does $n(t)$ approach as t tends to infinity? **1**

15. Given that

$$S = 1 - z + z^2 - z^3 + \ldots + z^{2n},$$

show that

$$S = \frac{1 + z^{2n+1}}{1+z}$$

where z is any complex number except -1. **2**

By putting $z = \cos\theta + i\sin\theta$, show that

$$1 - \cos\theta + \cos 2\theta - \cos 3\theta + \ldots + \cos 2n\theta = \frac{1 + \cos\theta + \cos 2n\theta + \cos(2n+1)\theta}{2(1+\cos\theta)}$$

provided $\cos\theta \neq -1$. **8**

16. A function f is defined by

$$f(x) = \frac{x^3}{x^2-1}, \quad x \neq \pm 1.$$

Find the local maxima and minima of f and obtain an equation for each of the asymptotes to the graph of f. Use this information to make a sketch of the graph of f. **8**

For a given real number k, the cubic equation

$$x^3 - k(x^2 - 1) = 0$$

can be rewritten in the form

$$\frac{x^3}{(x^2-1)} = k, \quad x \neq \pm 1.$$

Use this rearrangement and your graph of f to find the respective ranges of values of k for which the cubic equation has

(*a*) three real roots;

(*b*) only one real root. **3**

Marks

17. For $n = 0, 1, 2, \ldots$, let x_n and y_n denote the numbers of males and females respectively in a population of insects at the n^{th} generation.

Assume that the numbers x_n, y_n at the n^{th} generation are related to the corresponding numbers x_{n+1}, y_{n+1} at the $(n+1)^{\text{th}}$ generation by the matrix equation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} p & 0 \\ 1-p & 1 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad n = 0, 1, 2, \ldots$$

where p is a fixed real number such that $0 < p < 1$.

(*a*) Obtain expressions for x_{n+1} and y_{n+1} in terms of x_n, y_n and p. **1**

Deduce that the total size of the population remains constant from generation to generation. **2**

(*b*) Let $\mathbf{A} = \begin{pmatrix} p & 0 \\ 1-p & 1 \end{pmatrix}$.

Calculate $\mathbf{A}^2$ and $\mathbf{A}^3$ in terms of p, expressing your answers in as simple a form as possible. **2**

Hence deduce an expression for $\mathbf{A}^n$ in terms of p where n is a positive integer, and use this expression to verify that $\mathbf{A}^n\mathbf{A} = \mathbf{A}^{n+1}$ for all positive integers n. **2**

(*c*) Use (*b*) to show that, regardless of the initial numbers x_0 and y_0 of males and females, the males will approach extinction as the number of generations increases indefinitely. **2**

[END OF QUESTION PAPER]