Mathematics
Mathematics 1
Advanced Higher

6804

Spring 2000

HIGHER STILL

Mathematics

Mathematics 1
Advanced Higher

Support Materials

ACKNOWLEDGEMENTS

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MATHEMATICS 1 (ADVANCED HIGHER)

Introduction

These support materials for Mathematics were developed as part of the Higher Still Development Programme in response to needs identified at needs analysis meetings and national seminars.

Advice on learning and teaching may be found in *Achievement for All* (SOEID 1996), *Effective Learning and Teaching in Mathematics* (SOEID 1993), *Improving Mathematics* (SEED 1999) and in the Mathematics Subject Guide.

These notes are intended to support teachers/lecturers in the teaching of Mathematics 1 (AH). The resources referred to within the material are:

Understanding Pure Mathematics,
AJ Sadler and DWS Thorning, O.U.P., 1987 ISBN 0-19-914243-2

The complete A Level Mathematics,
Orlando Gough, Heinemann Educational Books, 1987, ISBN 0-435-51345-1

Mathematics In Action 6S,
John Hunter, Nelson Blackie, ISBN 0-17-441027-1

In these notes these texts are referred to as Sadler & Thorning, Orlando Gough and Hunter respectively.

ALGEBRA

CONTENT
know and use the notation $n\,!$, $^{n}C_{r}$, and $\begin{pmatrix} n \\ r \end{pmatrix}$

Comments

e.g Calculate $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$. Solve, for $n \in \mathbf{N}$, $\begin{pmatrix} n \\ 2 \end{pmatrix}$ =15.

This content is required for the Binomial expansion, it is unlikely to be assessed directly.

Students should be aware of the size of $n!$ for small values of n, and that calculator results consequently are often inaccurate (especially if the formula $^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$ is used). These results can be established numerically or from Pascal's triangle.

Teaching notes

Define $n\,! = n(n\text{-}1)(n\text{-}2) \ldots\ldots 3.2.1$ where $n \in \mathbf{W}$

e.g. $5\,! = 5.4.3.2.1 = 120$

(Note that $0\,! = 1$)

$$^{n}C_{r} = \frac{n!}{r!(n-r)!} \quad \text{where} \quad 0 \le r \le n$$

This is the number of ways of choosing r objects from n objects.

e.g. $$^{6}C_{2} = \frac{6!}{2!4!} = \frac{6\times5}{1\times2} = 15$$

e.g. $$\begin{pmatrix} 7 \\ 3 \end{pmatrix} = \frac{7!}{3!4!} = \frac{7\times6\times5}{1\times2\times3} = 35$$

Examples

1. 5 children are to sit on a bench in the gym. In how many ways can the children be seated !

   5.4.3.2.1 = 120

2. 2 children are chosen at random from the 5 children on the bench. How many different ways are there of choosing the children ?

$$^{5}C_{2} = \frac{5!}{2!3!} = \frac{5\times4}{1\times2} = 10$$

3. Find the value of n for which $\binom{n}{2}$ = 55s

$$\frac{n!}{2!(n-2)!} = 55$$

$$\frac{n(n-1)(n-2).......3.2.1}{2.(n-2)(n-3).......3.2.1} = 55$$

$$\frac{n(n-1)}{2} = 55$$

$n(n\text{-}1) = 110$

$n^2 - n - 110 = 0$

$(n - 11)(n + 10) = 0$

$n = 11$ or $n = - 10$ (not valid) $n = 11$

Exercises

1. Hunter	Page 34	Ex. 3.3 Q.1d-j, 2c,d, 6
2. Sadler and Thorning	Page 193	Ex. 7A Q.1,2,4,5,6,7,10
3. Orlando Gough	Page 239	Ex. 5.3:1 Q.2
	Page 242	Ex. 5.3:2 Q.1

CONTENT
know the results $\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ n-r \end{pmatrix}$ and $\begin{pmatrix} n \\ r-1 \end{pmatrix} + \begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n+1 \\ r \end{pmatrix}$

Comments

This content was contained in CSYS Paper 1 (unrevised).

Teaching notes

If we calculate $^n\mathrm{C}_r$ for $0 \le n \le 7$ and $0 \le r \le n$ and tabulate the results we obtain

		r							
		0	1	2	3	4	5	6	7
	0	1							
	1	1	1						
	2	1	2	1					
n	3	1	3	3	1				
	4	1	4	6	4	1			
	5	1	5	10	10	5	1		
	6	1	6	15	20	15	6	1	
	7	1	7	21	35	35	21	7	1

From the table we see that:

(a) each row is 'symmetrical', so $^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$

(b) each entry is the sum of two entries in the row above ie $^{n+1}\mathrm{C}_r = {}^n\mathrm{C}_{r-1} + {}^n\mathrm{C}_r$

These results may also be established algebraically using $^n\mathrm{C}_r = \dfrac{n!}{r!(n-r)!}$

$$\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ n-r \end{pmatrix}$$

LHS : $\quad = \dfrac{n!}{r!(n-r)!}$

RHS : $\quad = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{(n-r)!r!} = \dfrac{n!}{r!(n-r)!}$

LHS = RHS

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

$$\text{LHS : } \binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{r!(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}$$

$$= \frac{r.n!}{r(r-1)!(n-r+1)!} + \frac{(n-r+1)n!}{(n-r+1)(n-r)!r!}$$

$$= \frac{r.n!}{r!(n-r+1)!} + \frac{(n-r+1)n!}{r!(n-r+1)!}$$

$$= \frac{n!(r+n-r+1)}{r!(n-r+1)}$$

$$= \frac{(n+1)n!}{r!(n-r+1)!}$$

$$= \frac{(n+1)!}{r!(n-r+1)!}$$

$$\text{RHS : } \binom{n+1}{r} = \frac{(n+1)!}{r!(n+1-r)!}$$

$$= \frac{(n+1)!}{r!(n-r+1)!}$$

LHS = RHS

CONTENT
know Pascal's triangle

Comments

Pascal's triangle should be extended up to $n = 7$, however for assessment purposes $n \leq 5$. It is unlikely that this will be assessed directly.

Teaching notes

The array of numbers within the table in the previous section is known as **Pascal's triangle.** It is more commonly written in the following format

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

It is easily constructed and is a convenient means of evaluating $\begin{pmatrix} n \\ r \end{pmatrix}$

CONTENT
know and use the binomial theorem $(a+b)^n = \sum_{r=0}^{n} a^{n-r}b^r$, for $r, n \in \mathbf{N}$

Comments

For Binomial Theorem: e.g. expand $(x + 3)^4$, $(2u - 3v)^5$ [A/B]

This content was in CSYS Paper 1.

Teaching notes

The Binomial Theorem

Consider $(x+y)^n$

$(x+y)^1 = x+y$
$(x+y)^2 = x^2 + 2xy + y^2$
$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
…….

We note that the coefficients in the expansions can be found in Pascal's triangle.

e.g. $$(x+y)^4 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} x^4 + \begin{pmatrix} 4 \\ 1 \end{pmatrix} x^3y + \begin{pmatrix} 4 \\ 2 \end{pmatrix} x^2y^2 + \begin{pmatrix} 4 \\ 3 \end{pmatrix} xy^3 + \begin{pmatrix} 4 \\ 4 \end{pmatrix} y^4$$

For $(x+y)^n$ we have

$$\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \ldots . + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1}$$
$$+ \binom{n}{n}y^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}\,y^r$$

CONTENT
evaluate specific terms in a binomial expansion

Comments

E.g. find the term in x^7 in $(x+\frac{2}{x})^9$

Teaching notes

It is not necessary to write out the full expansion of the binomial theorem to evaluate a specific term.

e.g. to find the coefficient of x^7 in the expansion of $(x+\frac{2}{x})^9$

The general term is $\binom{9}{r}(x^2)^{9-r}(2x^{-1})^r \Rightarrow$ coefficient of $x^{9-2r} = \binom{9}{r}2^r$

$$\Rightarrow 9 - 2r = 7$$
$$\Rightarrow -2r = -2$$
$$\Rightarrow r = 1$$
$$\Rightarrow \text{coefficient of } x^7 = \binom{9}{1}2^1 = 18$$

Examples

1. $(2x-3)^5 = \binom{5}{0}(2x)^5(-3)^0 + \binom{5}{1}(2x)^4(-3)^1 + \binom{5}{2}(2x)^3(-3)^2 + \binom{5}{3}(2x)^2(-3)^3$
$+ \binom{5}{4}(2x)^1(-3)^4 + \binom{5}{5}(2x)^0(-3)^5$

$= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$

2. $(x^2 + \frac{1}{x})^3 = \binom{3}{0}(x^2)^3(x^{-1})^0 + \binom{3}{1}(x^2)^2(x^{-1})^1 + \binom{3}{2}(x^2)^1(x^{-1})^2 + \binom{3}{3}(x^2)^0(x^{-1})^3$

$$= x^6 + 3x^4x^{-1} + 3x^2x^{-2} + x^{-3}$$

$$= x^6 + 3x^3 + 3 + x^{-3}$$

Exercises

1.	Hunter	Page 35	Ex. 3.3 Q.11
2.	Orlando Gough	Page 251	Ex. 5.4:1 Q.1-14 (select questions)
3.	Sadler & Thorning	Page 226 Page 228	Ex. 8D (select) Ex. 8E

CONTENT
express a proper rational function as a sum of partial fractions (denominator of degree at most 3 and easily factorised)

Comments

E.g. express $\dfrac{5-10x}{1-3x-4x^2}$ in partial fractions.

This content was in CSYS Paper 1.

Teaching notes

The denominator may include a repeated linear factor or an irreducible quadratic factor. This is also required for integration of rational functions and is useful for graph sketching when asymptotes are present.

Partial Fractions

$\dfrac{1}{x-2} + \dfrac{3}{x+1}$ can be expressed as the single fraction $\dfrac{4x-5}{(x-2)(x+1)}$

To express $\dfrac{4x-5}{(x-2)(x+1)}$ as $\dfrac{1}{x-2} + \dfrac{3}{x+1}$ uses the process called **expressing in terms of partial fractions**

We assume that $\dfrac{4x-5}{(x-2)(x+1)} \equiv \dfrac{A}{x-2} + \dfrac{B}{x+1}$

where A and B are constants.

RHS: $\dfrac{A}{x-2} + \dfrac{B}{x+1} = \dfrac{A(x+1) + B(x-2)}{(x-2)(x+1)}$

LHS : $\frac{4x-5}{(x-2)(x+1)}$

Equating the numerators we have $4x - 5 = A(x + 1) + B(x - 2)$

There are now 2 methods for solving :

1. Choose suitable values for x to eliminate, in turn, A and B :

   $x = -1$ gives $-9 = B(-3)$ $B = 3$
   $x = 2$ gives $3 = 3A$ $A = 1$

2. Equate coefficients :
   x : $4 = A + B$
   Constants : $-5 = A - 2B$
   Solving simultaneously $A = 1$ and $B = 3$

Special cases

1. Denominator has a repeated linear factor

Example : express $\frac{4x}{(x-3)^2}$ as the sum of partial fractions

$$\frac{4x}{(x-3)^2} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} = \frac{A(x-3)}{(x-3)^2} + \frac{B}{(x-3)^2}$$

$4x = A(x - 3) + B$

$4x = Ax + B - 3\text{A}$

Equating coefficients :

Of x: $4 = A$
Constants : $0 = B - 3A = B - 12$
$B = 12$

$$\frac{4x}{(x-3)^2} = \frac{4}{(x-3)} + \frac{12}{(x-3)^2}$$

2. Denominator has an irreducible quadratic factor (i.e. it cannot be factorised)

Example : Express $\frac{5}{(x^2+1)(x-2)}$ as the sum of partial fractions

$$\frac{5}{(x^2+1)(x-2)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)}$$

$$= \frac{(Ax+B)(x-2)}{(x^2+1)(x-2)} + \frac{C(x^2+1)}{(x^2+1)(x-2)}$$

$$= \frac{(Ax+B)(x-2)+C(x^2+1)}{(x^2+1)(x-2)}$$

$$5 = (Ax+B)(x - 2) + C(x^2+1)$$

$$5 = Ax^2 + Cx^2 + Bx - 2Ax - 2B + C$$

Equating coefficients :

x^2 :	$0 = A + C$	(1)
x :	$0 = B - 2A$	(2)
constants :	$5 = C - 2B$	(3)
2 x (1)	$0 = 2A + 2C$	(4)
(2) + (4)	$0 = B + 2C$	(5)
2 x (5)	$0 = 2B + 4C$	(6)
(3) + (6)	$5 = 5C$	
	$C = 1$	
	$B = - 2$	
	$A = - 1$	

$$\frac{5}{(x^2+1)(x-2)} = \frac{-x-2}{(x^2+1)} + \frac{1}{(x-1)}$$

Exercises

1.	Hunter	Page 79	Q10 a,c,d,f,g (don't integrate)
2.	Orlando Gough	Page 98	Ex. 2.3:3 Q.1, 2, 3
3.	Sadler & Thorning	Page 454	Ex. 18A Q.1-18

CONTENT
include cases where an improper rational function is reduced to a polynomial and a proper rational function by division or otherwise [A/B]

Comments

E.g. express $\dfrac{x^3 + 2x^2 - 2x + 2}{(x-1)(x+3)}$ in partial fractions [A/B]

This content was in CSYS Paper 1.

Teaching notes

Divide the numerator by the denominator to obtain a polynomial and a proper rational function.

Express the proper rational function in terms of partial fractions.

Example

$$\frac{x^3}{(x-1)(x^2-2)}$$

$$\begin{array}{r|l} & \;1 \\ \hline x^3 - x^2 - 2x + 2 & x^3 \\ & \underline{x^3 - x^2 - 2x + 2} \\ & \qquad x^2 + 2x - 2 \end{array}$$

$$= 1 + \frac{x^2 + 2x - 2}{(x-1)(x^2-2)}$$

$$\frac{x^2 + 2x - 2}{(x-1)(x^2-2)} = \frac{A}{(x-1)} + \frac{Bx + C}{(x^2-2)} = \frac{A(x^2-2)}{(x-1)(x^2-2)} + \frac{(Bx+C)(x-1)}{(x^2-2)(x-1)}$$

$$x^2 + 2x - 2 = A(x^2\text{-}2) + (Bx + C)(x - 1)$$

$$= Ax^2 - 2A + Bx^2 + Cx - Bx - C$$

equating coefficients :

x^2 : $1 = A + B$ (1)
x : $2 = C - B$ (2)
constants : $-2 = -2A - C$ (3)

(1) + (2) $3 = A + C$ (4)

(3) +(4) $1 = -A$ hence $A = -1, C = 4, B = 2$

$$\frac{x^3}{(x-1)(x^2-2)} = 1 + \frac{-1}{(x-1)} + \frac{2x+4}{(x^2-2)}$$

Exercises

1.	Hunter	Page 79	Q10 b, e
2.	Orlando Gough	Page 98	Ex. 2.3:3 Q. 5,6
3.	Sadler & Thorning	Page 454	Ex. 18A Q.19 – 36

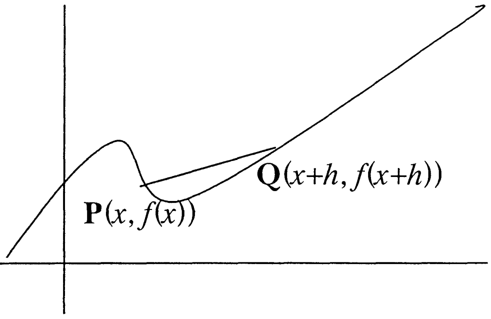
DIFFERENTIATION

CONTENT
know the meaning of the terms limit, derivative, differentiable at a point, differentiable on an interval, derived function, second derivative use the notation : $f'(x), f''(x), \frac{dy}{dx}, \frac{d^2y}{dx^2}$ recall the derivatives of x^α (α rational), sin$x$ and cos$x$

Comments

This content is unlikely to be assessed directly.

Teaching notes



The gradient of PQ is defined as $m_{PQ} = \frac{f(x+h) - f(x)}{h}$

and $f'(x) = \lim_{Q \to P} m_{PQ} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Students should be exposed to formal proofs of differentiation, although proofs will not be required for assessment purposes.

- $f'(x)$ is the derivative of $f(x)$ at x.
- Leibniz notation is $\frac{dy}{dx}$.
- $f'(x)$ is defined as the rate of change of f at x.
- $f(x)$ is differentiable on an interval I if $f'(x)$ exists for each $x \in I$.
- If the function $f'(x)$ is differentiable its derived function, $f''(x)$ is the second derivative of $f(x)$ at x.
- Leibniz notation is $\frac{d^2y}{dx^2}$.

Exercises

1.	Hunter	Page 63	Ex. 6.1 Q.1a-g, 2,3
2.	Orlando Gough	Page 54	Ex. 1.3:2 Q. 1,2,3
		Page 58	Ex.1.3:4 Q.1
3.	Sadler & Thorning	Page 261	Ex. 10B Q.1-61 (select)

CONTENT
know and use the rules for differentiating linear sums, products, quotients and composition of functions : $(f(x)+g(x))' = f'(x)+g'(x)$ $(kf(x))' = kf'(x)$, where k is a constant the chain rule : $(f(g(x))' = f'(g(x)).g'(x)$ the product rule : $(f(x).g(x))' = f'(x).g(x)+f(x).g'(x)$ the quotient rule : $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x).g(x)-f(x).g'(x)}{(g(x))^2}$ **differentiate given functions which require more than one application of one or more of the chain rule, product rule and the quotient rule [A/B]**

Comments

Students should be exposed to formal proofs although proofs will not be required for assessment. Computer Algebra Systems (CAS) may be used for consolidation/ extension. However when CAS are being used for difficult/real examples the emphasis should be on the understanding of concepts rather than routine computation. When software is used students should be able to say which rules were used.

This content was in CSYS Paper 1.

Teaching notes

Chain rule

This has been covered at Higher Grade.

Product rule

Let $y(x) = f(x).g(x)$

$y(x+h) = f(x+h).g(x+h)$

$$
\begin{aligned}
y'(x) &= \lim_{h\to 0}\frac{y(x+h)-y(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h).g(x+h)-f(x).g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h).g(x+h)-f(x+h).g(x)+f(x+h).g(x)-f(x).g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h).g(x+h)-f(x+h).g(x)}{h}+\lim_{h\to 0}\frac{f(x+h).g(x)-f(x).g(x)}{h}\\
&= \lim_{h\to 0}f(x+h).\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}g(x).\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= f(x).g'(x)+g(x).f'(x)
\end{aligned}
$$

Quotient rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x).g(x)-f(x).g'(x)}{(g(x))^2}$$

Before proving this, the derivative of $\left(\frac{1}{g(x)}\right)$ requires to be found.

$$\left(\frac{1}{g(x)}\right)' = \lim_{h\to 0}\frac{\frac{1}{g(x+h)}-\frac{1}{g(x)}}{h} = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{g(x+h)}-\frac{1}{g(x)}\right)$$

$$= \lim_{h\to 0}-\frac{1}{g(x)g(x+h)}\frac{g(x+h)-g(x)}{h} = -\frac{1}{[g(x)]^2}g'(x)$$

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x).\frac{1}{g(x)}\right)' = f(x)\left(\frac{1}{g(x)}\right)' + \frac{1}{g(x)}f'(x) \qquad \text{using Product Rule}$$

$$= f(x).-\frac{1}{[g(x)]^2}.g(x)+\frac{1}{[g(x)]^2}f'(x).g(x)$$

$$= \frac{f'(x).g(x)-f(x).g'(x)}{(g(x))^2}$$

Examples

1. $y = (x^2 + 3)^4$ (Chain rule) $\frac{dy}{dx} = 4(x^2+3)^3.2x = 8x(x^2+3)^3$

2. $y = (1 - x)^3(3x + 1)^2$ (Product rule)
   $f(x) = (1 - x)^3$ $f'(x) = -3(1-x)^2$
   $g(x) = (3x + 1)^2$ $g'(x) = 6(3x + 1)$

$$\frac{dy}{dx} = -3(1-x)^2.(3x+1)^2 + (1-x)^3.6(3x+1)$$

(take out common factor to simplify)

$$= 3(1 - x)^2(3x + 1)(1 - 5x)$$

3. $y = \dfrac{(x^2+1)^4}{x^3}$ (Quotient rule)

$f(x) = (x^2+1)^4 \quad f'(x) = 4(x^2+1)^3.2x$

$g(x) = x^3 \quad g'(x) = 3x^2$

$$\frac{dy}{dx} = \frac{8x(x^2+1)^3.x^3 - (x^2+1)^4.3x^2}{(x^3)^2}$$

(take out common factor in numerator and simplify)

$$= \frac{(x^2+1)^3(5x^2-3)}{x^4}$$

Exercises

1.	Hunter	Page 63	Ex. 6.1 Q.1h-y
2.	Orlando Gough	Page 67	Ex. 1.3:7 Q.1,2,8,10,11
3.	Sadler & Thorning	Page 261	Ex. 10B Q.54 - 59
		Page 330	Ex. 13E Q. 1-26 (select)

CONTENT
know the derivative of tan$x$ the definitions of sec x, cosec x and cot x the derivatives of e^x (exp x) and ln x know the definition $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ know the definition of higher derivatives $f^n(x), \frac{d^n y}{dx^n}$

Comments

Link the above derivatives with the graphs of the functions.
The definitions of e^x and ln x should be revised and examples given of their occurrence.

Students should be aware that not all functions are differentiable everywhere, e.g. $f(x) = |x|$ at $x = 0$. Use of software is recommended.

Students should also know that higher derivatives can have discontinuities and be aware of the graphical effects of this, i.e. lack of smoothness.

e.g. $f(x) = \begin{cases} -x^2, x < 0 \\ x^2, x \geq 0 \end{cases}$ for which f and f' are continuous but f'' is not.

Most of this content was in CSYS Paper 1, higher derivative was contained in CSYS Paper 2 (unrevised).

Teaching notes

Derivative of tan$x$

Let $y = \tan x = \dfrac{\sin x}{\cos x}$ (Quotient rule), $f(x) = \sin x$ and $g(x) = \cos x$

$$\frac{dy}{dx} = \frac{f'(x).g(x) + f(x).g'(x)}{(g(x))^2} = \frac{\cos x.\cos x - \sin x.(-\sin x)}{(\cos x)^2} = \frac{1}{\cos x} = \sec^2 x$$

Note : the derivative is not defined for $x = (2n+1)\frac{\pi}{2}$, $n \geq 0$

Students will require to be introduced to the following functions:

$\sec x = \dfrac{1}{\cos x}$, $\operatorname{cosec} x = \dfrac{1}{\sin x}$ and $\cot x = \dfrac{1}{\tan x}$

Derivatives of e^x and ln x

Define $y = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.. = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots..$

Differentiate term by term to give:

$$\frac{dy}{dx} = 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + \ldots\ldots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots.. = e^x$$

If $y = \ln x$ then $e^y = x$ (inverse functions)

Differentiate with respect to y $\Rightarrow e^y = \dfrac{dx}{dy} \Rightarrow \dfrac{1}{e^y} = \dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x}$

i.e. the derivative of $\ln x$ is $\dfrac{1}{x}$.

Examples

1. $f(x) = \tan 3x$ find $f'(x)$. (Chain rule)

   $f'(x) = \sec^2 3x \times 3 = 3\sec^2 3x$

2. $f(x) = \ln (x^2 - 1)$ find $f'(x)$. (Chain rule)

   $$f'(x) = \frac{1}{x^2 - 1} 2x = \frac{2x}{x^2 - 1}$$

3. $f(x) = e^{\sin x}$ find $f'(x)$. (Chain rule)

   $f'(x) = e^{\sin x} \cos x = \cos x \, e^{\sin x}$

4. $h(x) = x^4 e^x$ find $h'(x)$. (Product rule)

   $f(x) = x^4 \quad f'(x) = 4x^3 \qquad g(x) = e^x \quad g'(x) = e^x$

   $$\begin{aligned} h'(x) &= f'(x).g(x) + f(x).g'(x) = 4x^3 e^x + x^4 e^x \\ &= x^3 e^x (4 + x) \end{aligned}$$

5. $$k(x) = \ln \frac{2x+1}{x-1}$$

   For the derivative of $\frac{2x+1}{x-1}$ use the quotient rule

   $f(x) = 2x + 1 \quad f'(x) = 2 \qquad g(x) = x - 1 \quad g'(x) = 1$

   $$\frac{f'(x).g(x) - f(x).g'(x)}{(g(x))^2} = \frac{2(x-1) - (2x+1)}{(x-1)^2} = \frac{-3}{(x-1)^2}$$

   Using the chain rule :

   $$k'(x) = \frac{1}{2x+1\big/_{x-1}} \frac{-3}{(x-1)^2} = \frac{-3}{(2x+1)(x-1)}$$

6. $y = 3x^3 - 4x^2 + 5x - 2$

   $$\frac{dy}{dx} = 9x^2 - 8x + 5$$

   $$\frac{d^2y}{dx^2} = 18x - 8$$

$$\frac{d^3y}{dx^3} = 18$$

Exercises

1.	Hunter	Page 69	Ex. 6.2 Q.1a,b,c,d,e,f,g, h,k,l,m, 2
2.	Orlando Gough	Page 58	Ex. 1.3:4 Q.1-4
		Page 105	Ex. 2.5:2 Q. 5,6,7
		Page 108	Ex. 2.5:4 Q. 8,10,11
		Page 160	Ex. 3.4:3 Q.10i,ii,vii,x 13i,iia,c, 14,15,16ia,b,d,e
3.	Sadler & Thorning	Page 370	Ex.15F Q.24, 33, 35 – 46

CONTENT
apply differentiation to : a) rectilinear motion b) extrema of functions : the maximum and minimum values of a continuous function *f* defined on a closed interval [a,b] can occur at stationary points, end points or **points where f' is not defined [A/B]** c) optimisation problems

Comments

Rectilinear motion : e.g. find the acceleration of a particle whose displacement s metres from a certain point at time t seconds is given by $s = 8 - 75t + t^3$

Extrema of functions: e.g. Find maximum value of the function

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \end{cases} \qquad \text{[A/B]}$$

This content was in CSYS Paper 1, although the example given above could be described as new.

Teaching notes

Rectilinear motion

The velocity v of a body is defined as the rate of change of the displacement s of that body from a fixed origin, with respect to time t.

i.e $v = \dfrac{ds}{dt}$

The acceleration a of a body is defined as the rate of change of the velocity v of that body with respect to time t.

i.e. $a = \frac{dv}{dt}$

Examples

1. A pebble is thrown vertically upwards from a point O. The height of the stone, s metres, above O, after t seconds is given by $s = 8t - 2t^2$.

   a) Calculate the velocity of the pebble at the point at which it is thrown.

   $$v = \frac{ds}{dt} = 8 - 2t$$

   At the point at which it is thrown, $t = 0$ $v = 8$. The velocity is 8 m/s.

   b) Calculate the acceleration of the pebble as it lands.

   At the point at which it lands $s = 0$

   $$8t - 2t^2 = 0$$
   $$2t(4 - t) = 0$$
   $$t = 0 \quad \text{or} \quad t = 4$$

   $a = \frac{dv}{dt} = -2$ The acceleration is -2m/s^2 i.e. decceleration of 2 m/s^2

2. The displacement s of point P at time t is given by $s = 4t^3 - t^2 + 20t - 10$

a) Find the velocity of P after 5 seconds.

$v = \frac{ds}{dt} = 12t^2 - 2t + 20$ $t = 5$ $v = 310$ Velocity = 310m/s

b) Find the time at which the acceleration of P is 22m/s^2

$a = \frac{dv}{dt} = 24t - 2$, $24t - 2 = 22$, $24t = 24$, $t = 1$ After 1 second.

Exercises

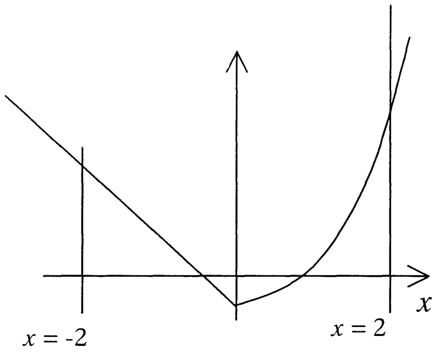
1.	Hunter	Page 92	Ex. 8.1 Q.2
2.	Orlando Gough	Page 370	Ex. 7.1:1 Q.1-16 (select)
3.	Sadler & Thorning	Page 312	Ex.12D Q. 1-8 (select)

Maximum/minimum values

The concept of maximum and minimum values of a continuous function at end points or stationary points has been covered at Higher Grade. What has not been covered is the idea of a maximum/minimum value occurring at a point where f' is not defined.

Example

$$f(x) = \begin{cases} -x, & -2 \leq x \leq 0 \\ x^2, & 0 \leq x \leq 2 \end{cases}$$



In this case we have a continuous graph but $f'(x)$ is not defined at $x = 0$.

Maximum value = $2^2 = 4$
Minimum value = 0

Exercises

1.	Orlando Gough	Page 61	Ex. 1.3:5 Q.7
2.	Sadler & Thorning	Page 281	Ex.11A Q.6

(Note – there are virtually no questions on this particular aspect of the course)

Optimisation

This has been covered in Higher Grade, but should be extended by using the rules of differentiation contained in this unit.

Exercises

1.	Orlando Gough	Page 62	Ex. 1.3:5 Q.14-28 (select)
2.	Sadler & Thorning	Page 269	Ex.10D Q.21-25

INTEGRATION

CONTENT
know the meaning of the terms integrate, integrable, indefinite integral, definite integral and constant of integration
recall standard integrals of x^{α} $(\alpha \in Q, \alpha \neq -1)$, sinx and cosx
$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$, $a,b \in$ **R**
$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad a < c < b$
$\int_a^b f(x)dx = -\int_b^a f(x)dx, \quad b \neq a$
$\int_a^b f(x)dx = F(b) - F(a)$, where $F'(x) = f(x)$
know the integrals of e^x , x^{-1} , $\sec^2 x$

Comments

Nearly all of this content has been covered at Higher level. This content is unlikely to be assessed directly.

Computer Algebra Systems (CAS) could be used for consolidation/extension. When CAS are being used the emphasis should be on understanding of the concepts rather than routine computation. Students should be able to identify the rules being used.

Teaching notes

Using the facts that :

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\int e^x dx = e^x + C \qquad \int \frac{1}{x}dx = \ln x + C \qquad \int \sec^2 x dx = \tan x + C$$

Examples

1. $\int 4e^{2x} dx = \frac{1}{2}4e^{2x} + C = 2e^{2x} + C$

2. $\int \frac{1}{3x-2}dx = \frac{1}{3}\ln(3x-2) + C = \frac{1}{3}\ln(3x-2) + C$

3. $\int \sec^2 4x dx = \frac{1}{4}\tan 4x + C = \frac{1}{4}\tan 4x + C$

4. $\int \frac{2x+1}{x-1} dx$ Before integrating, divide the denominator into the numerator.

$$\begin{array}{r} 2 \\ x-1\overline{)2x+1} \\ \underline{2x-2} \\ 3 \end{array} \qquad \text{i.e. } \frac{2x+1}{x-1} = 2 + \frac{3}{x-1}$$

$$\int \frac{2x+1}{x-1} dx = \int 2 + \frac{3}{x-1} dx = 2x + 3\ln(x-1) + C$$

5. $\int \frac{x+3}{x^2+3x+2} dx$ Before integrating, this must be rewritten using partial fractions

$$\frac{x+3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

i.e. $x + 3 = A(x + 2) + B(x + 1)$

For $x = -2 \quad 1 = -B \Rightarrow B = -1$
For $x = -1 \quad 2 = A \Rightarrow A = 2$

$$\int \frac{x+3}{x^2+3x+2} dx = \int \frac{2}{x+1} dx - \int \frac{1}{x+2} dx$$

$$= 2\ln(x+1) - \ln(x+2) + C$$

$$= \ln(x+1)^2 - \ln(x+2) + C$$

$$= \ln\frac{(x+1)^2}{(x+2)} + C$$

Exercises

1. Hunter	Page 76	Ex. 7.1 Q. 1a,b,c,d,e,f,j,m,n,s,t,
	Page 85	Ex. 7.2 Q. 1a,b,,d,e,f,g,h
2. Orlando Gough	Page 171	Ex. 4.1:2 Q.1,2,15,18,20,
	Page 176	Ex. 4.1:3 Q.1,2,3,4
3. Sadler & Thorning	Page 296	Ex.12A Q. 2-15 (select)
	Page 480	Ex. 19A Q.16-27 (select)
	Page 487	Ex. 19B Q. 26-28, 34-36 39-41, 43a,b,e,f, 46, 48

CONTENT
integrate by substitution : expression requiring a simple substitution expressions where the substitution will be given (e.g. $\int \cos^3 x \sin x \, dx,\ u = \cos x$) the following special cases of substitution $\int f(ax + b)\, dx$, (eg. $\int \sin(3x + 2)\, dx$) $\int \frac{f'(x)}{f(x)} dx$, (e.g. $\int \frac{2x}{x^2+3} dx$

Comments

Students are expected to integrate simple functions on sight e.g. $\int x\, e^{x^2} dx$.
This content was in CSYS Paper 1.

Teaching notes

- Look at the expression carefully to 'separate' it into a function and its' derivative.
- Look out for when the integral can be expressed in the form :

$$\frac{\text{k} \times \text{'derivative'}}{\text{function}} \Rightarrow \int \frac{\text{k} \times \text{'derivative'}}{\text{function}} = \text{k} \times \ln(\text{function}) + C$$

Examples

1. $\int 2x \sin(x^2)\, dx$: $2x$ is the derivative of $x^2 \Rightarrow$ let $u = x^2$

$$\frac{du}{dx} = 2x$$
$$du = 2x\, dx$$

$$\sin u = \sin(x^2)$$

$$\int 2x \sin(x^2)\, dx = \int \sin u\, du = -\cos u + C$$

$$= -\cos(x^2) + C$$

2. $\int x^2 e^{x^3} dx$: $3x^2$ is the derivative of $x^3 \Rightarrow$ let $u = x^3$

$$\frac{du}{dx} = 3x^2$$
$$du = 3x^2\, dx$$

$$e^{x^3} = e^u$$

$$\int x^2 e^{x^3} dx = \int \frac{1}{3} 3x^2 e^{x^3} dx = \int \frac{1}{3} e^u du = \frac{1}{3} e^{x^3} + C$$

3. Using the substitution $u = \sin x$ find $\int \cos x\,(\sin x)^{½}\,dx$

$$u = \sin x \quad \frac{du}{dx} = \cos x \quad du = \cos x dx \quad (\sin x)^{½} = u^{½}$$

$$\begin{aligned}\int \cos x\,(\sin x)^{½}\,dx &= \int (\sin x)^{½} \cos x\,dx \\ &= \int u^{½}\,du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}(\sin x)^{3/2} + C\end{aligned}$$

4. Evaluate $\int_0^1 \frac{2x}{(x^2+3)}dx$ Let $u = x^2 + 3$

$$\frac{du}{dx} = 2x$$
$$du = 2xdx$$

$$\int_0^1 \frac{2x}{(x^2+3)}dx = \int \frac{1}{u}du = \ln u + C = \Big[\ln(x^2+3)\Big]_0^1 = \ln 4 - \ln 3 = \ln\frac{4}{3}.$$

5. $\int \frac{x^2+1}{x^3+3x-5}dx$ Let $u = x^3 + 3x - 5$

$$\frac{du}{dx} = 3x^2 + 3$$
$$du = (3x^2 + 3)dx$$

$$\frac{1}{3}du = (x^2 + 1)dx$$

$$\int \frac{x^2+1}{x^3+3x-5}dx = \int \frac{1}{3u}du = \frac{1}{3}\ln u + C = \frac{1}{3}\ln(x^3+3x-5) + C$$

Exercises

1. Hunter	Page 77	Ex. 7.1 Q. 3a,b,c,,j,k
2. Orlando Gough	Page 186	Ex. 4.2:1 Q.1a,b,f,g,h,I,j 2vi,vii,7i,iv,vii,viii
	Page 190	Ex. 4.2:2 Q.5,6,7iii,v,8,12

3. Sadler & Thorning	Page 500	Ex.20A Q. 1,3,5,7,9,
	Page 508	Ex.20C Q.10,11,15,26,35, 42, 50,52,56

CONTENT
use an elementary treatment of the integral as a limit using rectangles apply integration to the evaluation of areas **including integration with respect to y [A/B]**

Comments

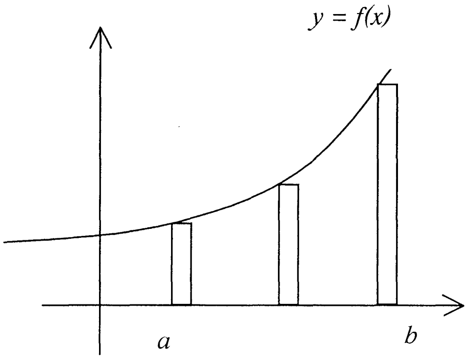
Other applications may include :

(i) volume of simple solids of revolution (disc/washer method) [A/B]

(ii) speed/time graph [A/B]

Teaching notes

Consider the area bounded by the curve $y = f(x)$, the x axis from $x = a$ to $x = b$.



An estimate of this area is the sum of the rectangular strips of equal width which the area can be approximately divided into. The greater the number of strips the closer the estimate.

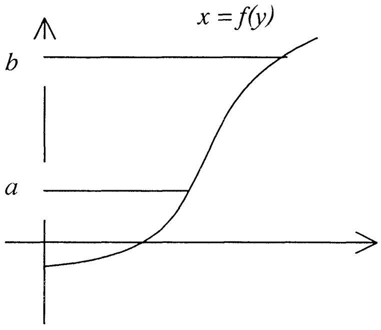
ie. The narrower the width of each strip the closer the estimate.

The estimate of this area $= \lim\limits_{dx \to 0} \sum\limits_{x=a}^{b} f(x)dx$

As a limit is taken $dx \to 0$ **Area** $= \int_a^b f(x)dx$

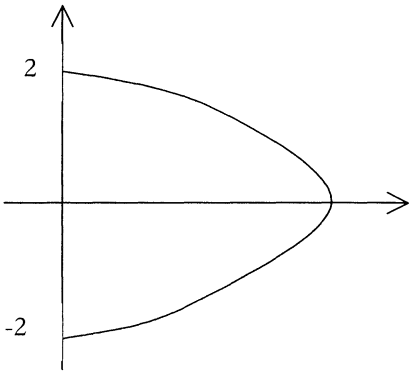
Area between a curve and the y-axis

In this cases we are finding the area under $x = f(y)$ and the y axis bounded by $y = a$ and $y = b$.



$$\text{Area} = \int_a^b f(y)dy$$

Example



Find the area enclosed by the curve $y^2 = 4 - x$ and the y axis.

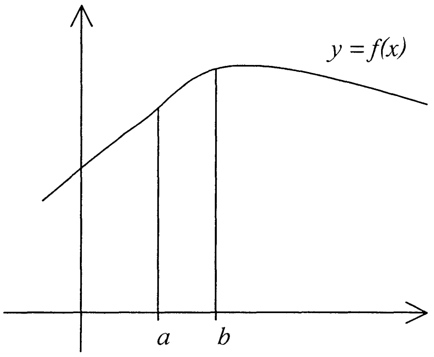
$y^2 = 4 - x \Rightarrow x = 4 - y^2$

$y^2 = 4 - x$ intersects the y axis when $x = 0$
$y^2 = 4$
$y = \pm\ 2$

$$\text{Area} = \int_{-2}^{2} (4 - y^2)\, dy = \left[4y - \frac{1}{3}y^3\right]_{-2}^{2} = (8 - \frac{8}{3}) - (-8 + \frac{8}{3})$$

$$\text{Area} = \frac{32}{3} \text{ units }^2$$

Volume of a solid of revolution



Consider the area under $y = f(x)$ bounded by the x axis between $x = a$ and $x = b$ as shown in the diagram.

If it is rotated through one revolution about the x axis the volume generated is given by :

$$\text{Volume} = \pi \int_a^b y^2 \ dx$$

Example

Find the volume of the solid formed when the area under $y = 3$ between $x = 1$ and $x = 2$ is rotated through one revolution about the x axis.

$$V = \pi \int_1^2 3^2\, dx = \pi\ [9x\ ]_1^2 = \pi\ (18 - 9)$$

Volume = $9\ \pi$ units ^{3}

Note : This will be the volume of a cylinder with radius = 3 units, height =1 unit

Exercises

1. Hunter	Page 87	Ex. 7.2 Q. 10 (volume)
2. Orlando Gough	Page 181	Ex. 4.1:4 Q.1,2,3,4,5,6 (volumes)
3. Sadler & Thorning	Page 304	Ex.12B Q.15,16 (areas)
	Page 307	Ex.12C Q. 1-8 (volumes)

Applications

We already know the following relationships between time (t) distance (s), velocity (v) and acceleration (a) :

$$v = \frac{ds}{dt} \text{ and } a = \frac{dv}{dt}$$

Consequently : $s = \int vdt$ and $v = \int adt$

Examples

1. A particle moves in straight line from an origin. At time t seconds its acceleration is given by :

$a = 12t - 1$

When $t = 1$ the velocity of the particle is 8m/s and its displacement is 4 metres. Find expressions for the velocity and displacement of the particle in terms of t.

$v = \int adt = \int (12t - 1)\, dt$, i.e. $v = 6t^2 - t + C$

Using $v = 8$ and $t = 1$ $8 = 6.1^2 - 1 + C$

$8 = 5 + C$

$C = 3$

$v = 6t^2 - t + 3$

$s = \int vdt = \int (6t^2 - t + 3)\, dt$ $s = t^3 - \frac{1}{2}t^2 + 3t + C$

Using $s = 4$ and $t = 1$ $4 = 1 - \frac{1}{2} + 3 + C$

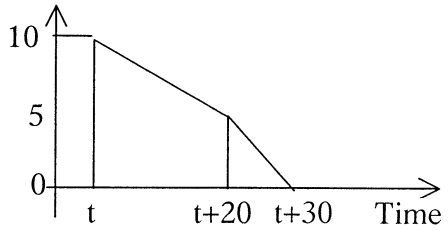
$4 = 3\frac{1}{2} + C$

$C = \frac{1}{2}$

$s = t^3 - \frac{1}{2}t^2 + 3t + \frac{1}{2}$

2. A cyclist is travelling at a uniform speed of 10 m/s when he starts to decelerate. After 20 seconds of deceleration his speed is 5 m/s. He comes to rest after a further 10 seconds, decelerating being uniform over the 10 seconds.

Find the deceleration over the 2 intervals and the total distance travelled over the 30 seconds.



Since acceleration = $\frac{dv}{dt}$ = gradient = $\frac{5-10}{20} = -\frac{5}{20}$

deceleration = $-\frac{1}{4}$ m/s^2 over the first interval

Over the second interval gradient = $\frac{0-5}{10} = -\frac{1}{2}$

deceleration = $\frac{1}{2}$ m/s

Total distance travelled = area under the graph

(since $s = \int v\,dt$) = (20 x 5) + ½(20 x 5) + ½(10 x 5)

= 175 metres

Exercises

1. Hunter	Page 92	Ex. 8.1 Q. 2,4,7
2. Orlando Gough	Page 392	Ex. 7.2:2 Q.1-7 (Uses position vectors)
3. Sadler & Thorning	Page 312	Ex. 12D Q.1-8

PROPERTIES OF FUNCTIONS

CONTENT
know the meaning of the terms functions, domain, range, inverse function, critical point, stationary point, point of inflection, concavity, local maxima and minima, global maxima and minima, continuous, discontinuous, asymptote
determine the domain and range of a function
use the derivative test for locating and identifying stationary points

Comments

An appropriate derivative test would be

Concave up $\Leftrightarrow$ $f''(x) > 0$;

Concave down $\Leftrightarrow$ $f''(x) < 0$;

Note that a necessary and sufficient condition for a point of inflection is a change in concavity.

Care should be exercised when using the second derivative test in preference to the first derivative test. The second derivative may not exist, and even when it does an can easily be computed, it may not be helpful, e.g. $f(x) = x^4$ at $x = 0$. Here $f''(0) = 0$ which is inconclusive. The first derivative test, however, easily shows there is a local minimum at $x = 0$.

This content was in CSYS Paper 1.

Teaching notes

Many of the terms used here will be familiar from Higher level. Those possibility not covered will be :

Concavity

Concavity is an indication of the shape of the curve. There are two possibilities :

Concave up :

Concave up :

Or

Concave down :

Concave down :

It is also true that where a curve is concave up then $f''(x) > 0$ and where it is concave down then $f''(x) < 0$. This can be used for locating and identifying stationary points. Where critical points have been identified ie. $f'(a) = 0$ then for $f''(a) > 0$ then $(a, f(a))$ is a local minimum and when $f''(a) < 0$ then $(a, f(a))$ is a local maximum.

Where concavity changes from up to down or from down to up in moving from left to right through point $(a, f(a))$ then the point $(a, f(a))$ is a point of inflection. Consequently $f''(a) = 0$.
Note : where $f''(a) = 0$ and $f'(a) \neq 0$ this is a **horizontal** point of inflection as opposed to a **sloping** point of inflection where $f'(a) = 0$.

Continuous/discontinuous

Function $y = f(x)$ is defined as being continuous at $x = a$ if :

$$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$$

i.e. if you can draw the graph of $y = f(x)$ near $x = a$ without taking your pen off the paper.

Function $y = f(x)$ is defined as being discontinuous at $x = a$ if it is not continuous at $x = a$.

Global maximum/minimum

The global maximum is the largest value of the function attained in an interval and therefore, cannot be less than any value on that interval. A *local* maximum can be less than other values, even a local minimum.

Asymptotes

An asymptote is a line which can be considered as a tangent to the curve $y = f(x)$ as x approaches $\pm\infty$.
We will consider the different types of asymptotes later.

Example

Find the stationary points of the function given by $y = x^3 - 6x^2 + 5$ and state their nature.

$$\frac{dy}{dx} = 3x^2 - 12x = 3x(x-4) \qquad \text{For } \frac{dy}{dx} = 0 \quad 3x(x-4) = 0 \Rightarrow \quad x = 0 \quad \text{or } x = 4$$

$$\frac{d^2y}{dx^2} = 6x - 12 \qquad \text{For } \frac{d^2y}{dx^2} = 0 \quad 6x - 12 = 0 \Rightarrow x = 2$$

When $x = 0$ $y = 5$, $x = 4$ $y = -27$, $x = 2$ $y = -11$

$$x = 0 \Rightarrow \frac{d^2y}{dx^2} = -12 \Rightarrow \text{concave down, i.e. local maximum at (0, 5)}$$

$$x = 4 \Rightarrow \frac{d^2y}{dx^2} = 12 \Rightarrow \text{concave up, i.e. local minimum (4, -27)}$$

$$x = 2 \Rightarrow \frac{d^2y}{dx^2} = 0 \text{ and } \frac{dy}{dx} = -12 \Rightarrow \text{horizontal point of inflexion at (2, -11)}$$

Exercises

1. Hunter	Page 70	Ex. 6.2 Q. 5a-d,6a-d,8
2. Orlando Gough	Page 31	Ex. 1.2:3 Q.1-32 (domain & range - select questions)
	Page 42	Ex.1.2:5 Q.1-4(continuous /discontinuous – select questions)
	Page 61	Ex.1.3:5 Q.1-9 (stationary points – select and don't sketch)
3. Sadler & Thorning	Page 32	Ex. 1A Q. 1-14 (domain, range & inverse functions – select questions)
	Page 268	Ex. 10D Q. 1-16 (stationary points – select questions)
	Page 273	Ex. 10F Q. 1-9 (points of inflection – select questions)

CONTENT
sketch the graphs of $\sin x$, $\cos x$, $\tan x$, e^x, $\ln x$ and their inverse functions, simple polynomial functions

Comments

This content was in CSYS Paper 1.

Teaching notes

This has all been covered at Higher level but may need some revision.

Exercises

1. Hunter	Page 56	Ex. 5.2 Q. 1

CONTENT
know and use the relationship between the graph of $y = f(x)$ and the graphs of : $y = kf(x)$ $y = f(x) + k$ $y = f(x + k)$ $y = f(kx)$ know and use the relationship between the graph of $y = f(x)$ and the graphs of : $y = \|f(x)\|$ $y = f^{-1}(x)$ given the graph of a function f, sketch the graph of a related function

Comments

Inverses should be considered as reflection in the line $y = x$. Care must be taken over the domain and range when finding inverses.

For the sketching of rational functions, the degree of both the numerator and denominator will be less than or equal to two.

Teaching notes

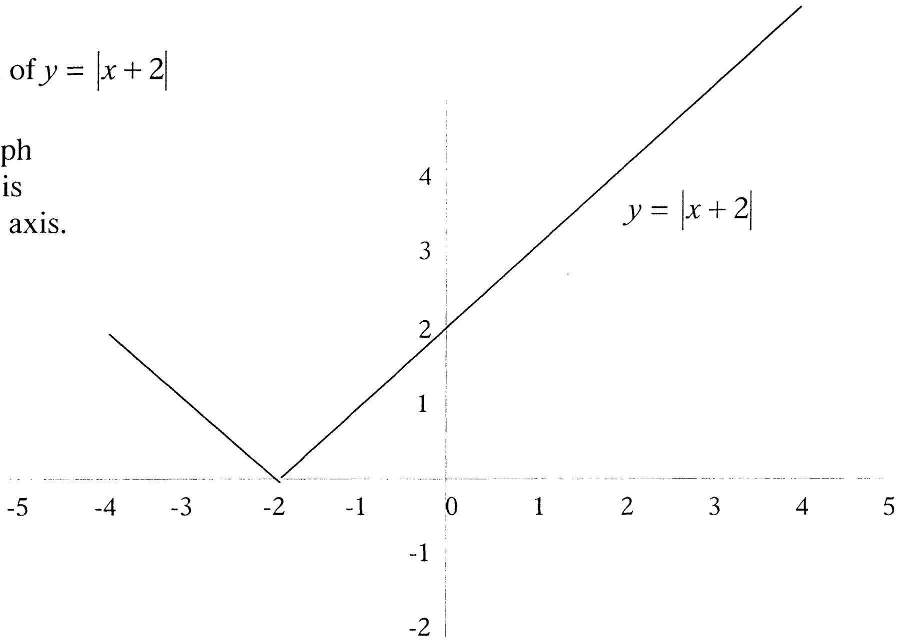
All of the first section of content has been covered at Higher level and possibly the second. It is worth reminding students of the relationship between the graphs of $y = f(x)$ and $y = |f(x)|$

To obtain the graph of $y = |f(x)|$ firstly sketch that of $y = f(x)$. Retain those parts of the graph for which y is positive and for those parts for which y is negative, reflect the graph in the x axis.

Example

Sketch the graph of $y = |x + 2|$

Note that the graph below the x axis is reflected in the x axis.



Exercises

1. Orlando Gough	Page 105 Ex. 2.5:2 Q. 3 (exp) Page 106 Ex. 2.5:3 Q. 1,2 (inv) Page 108 Ex. 2.5:4 Q. 5.6 (ln*x*) Page 128 Ex. 3.1:2 Q.7-12 (trig) Page 99 Ex. 2.4:1 Q.1,2 (modulus)
2. Sadler & Thorning	Page 108 Ex.4B Q. 1-8 (trig-select) Page 281 Ex. 11A Q. 4b,d (modulus) Page 45 Ex. 1D Q. 5

CONTENT
determine whether a function is even or odd or neither and symmetrical and use these properties in graph sketching

Teaching notes

A function $y = f(x)$, defined on the set **R**, is **even** if $f(-x) = f(x)$ and it is **odd** if $f(-x) = -f(x)$

e.g. $y = x^2$ is an even function, $y = x^3$ is an odd function
e.g. $y = \cos x$ is an even function, $y = \sin x$ is an odd function

Exercises

1. Orlando Gough Page 38 Ex. 1.2:4 Q. 21,22,23
2. Sadler & Thorning Page 45 Ex.1D Q. 1-4

CONTENT
sketch graphs of real functions using available information, derived from calculus and/or algebraic arguments, on zeros, asymptotes (vertical and non-vertical), critical points, symmetry

Comments

This content was in CSYS Paper 1.

Teaching notes

Asymptotes

Asymptotes can be considered as tangents to a curve as x approaches $\pm\infty$.

Vertical asymptotes

Let $y = \dfrac{F(x)}{f(x)}$

If $f(x)$ can be factorised into linear factors e.g. $(x - a)$ and $(x - b)$ then y is not defined for $x = a$ or $x = b$ and the lines $x = a$ and $x = b$ are vertical asymptotes. For values of x very close to a and b, y will be extremely large, either positive or negative. This gives us the **approaches to the asymptote.**

Horizontal/slant asymptotes

Let $y = \dfrac{F(x)}{f(x)}$

There are three possibilities :

i) If $F(x)$ is of degree <u>less</u> than $f(x)$ then $y \to 0$ as $x \to \pm\infty$ and so $y = 0$ is an asymptote.
For **approaches to the asymptote** we consider whether y is positive or negative as $x \to -\infty$ and $x \to +\infty$.

ii) If $F(x)$ is of the <u>same</u> degree as $f(x)$ then, by division,
$y = k + \dfrac{\text{remainder}}{f(x)}$, where k is a constant
so $y \to k$ as $x \to \pm\infty$, thus $y = k$ is an asymptote.
For **approaches to the asymptote** we consider whether $\dfrac{\text{remainder}}{f(x)}$ is positive or negative as $x \to -\infty$ and $x \to +\infty$.
.

iii) If $F(x)$ is of degree **1 higher** than $f(x)$ then, by division,
$y = ax + b + \dfrac{\text{remainder}}{f(x)}$
so $y \to ax + b$ as $x \to \pm\infty$ and thus $y = ax + b$ is an asymptote.
For **approaches to the asymptote** we consider whether $\dfrac{\text{remainder}}{f(x)}$ is positive or negative as $x \to -\infty$ and $x \to +\infty$

Example

Sketch $y = \dfrac{x^2}{x-1}$ showing clearly any critical points and show how the curve approaches its asymptotes.

Using Quotient rule:

$$\frac{dy}{dx} = \frac{2x(x-1)-x^2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2} = 0 \Leftrightarrow x = 0, x = 2$$

$$\frac{d^2y}{dx^2} = \frac{(2x-2)(x-1)^2 - 2(x-1)(x^2-2x)}{(x-1)^4} = \frac{(2x-2)(x-1) - 2(x^2-2x)}{(x-1)^3} = \frac{2}{(x-1)^3}$$

There are no values for which $\dfrac{d^2y}{dx^2} = 0$ so no horizontal points of inflexion.

$x = 0 \Rightarrow \dfrac{d^2y}{dx^2} = -2 \Rightarrow$ maximum turning point at (0, 0)

$x = 2 \Rightarrow \dfrac{d^2y}{dx^2} = 2 \Rightarrow$ minimum turning point at (2, 4)

Vertical asymptote: $x = 1$ from the linear factor $(x - 1)$ As $x \to 1^- \; y \to -\infty$
$x \to 1^+ \; y \to +\infty$

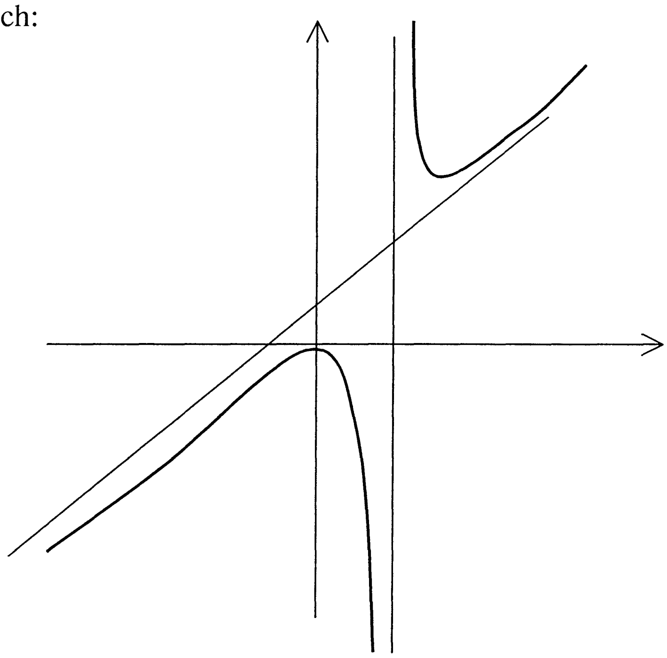
Slant asymptote: Dividing: $y = x + 1 + \dfrac{1}{x-1}$. As x $\to \pm\infty, y \to x + 1$

Thus $y = x + 1$ is an asymptote

As $x \to -\infty$, $\dfrac{1}{x-1} \to 0^-$ and so $y \to (x + 1)^-$

As $x \to +\infty$, $\dfrac{1}{x-1} \to 0^+$ and so $y \to (x + 1)^+$

Sketch:



Exercises

1. Hunter	Page 56	Ex. 5.2 Q. 3,4
2. Orlando Gsough	Page 93	Ex. 12.3:1 Q.1,2,5,6,7,12,18,19,20
3. Sadler & Thorning	Page 342	Ex. 14A Q.3,4, 9-19 (select)

Note: There are no examples which include oblique asymptotes on Orlando Gough.

SYSTEMS OF LINEAR EQUATIONS

CONTENT
use elementary row operations (EROs)
reduce to upper triangular form using EROs
solve a 3x3 system of linear equations using Gaussian elimination on an augmented matrix
find the solution of a system of linear equations Ax = B, where A is a square matrix, include cases of unique solutions, **no solution (inconsistency) and an infinite family of solutions A/B**
know the meaning of the term ill-conditioned A/B
compare the solutions of related systems of two equations in two unknowns and recognise ill-conditioning A/B

Comments

For Gaussian elimination only 3x3 cases are required for assessment purposes, and at grade C they are restricted to those with a unique solution. Larger systems can be tackled using computer packages or advanced calculators.

Ill-conditioning can be introduced by comparing the solutions of the following systems:

a) $x + 0.99y = 1.99$ b) $x + 0.99y = 2.00$

$0.99x + 0.98y = 1.97$ $0.99x + 0.98y = 1.97$

Some of this content was in CSYS Paper 1, however the majority of the content is from Paper 4.

Teaching notes

An array of rows and columns is known as a matrix. The number of rows and columns determines the **order** of the matrix. A matrix with 3 rows and 4 columns is said to be a 3 x 4 matrix, i.e. it is a matrix of order 3 x 4. Each entry in the matrix is called an **element**.

Consider $A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \\ -3 & 4 \end{pmatrix}$

A is a 2 x 3 matrix where –1 is the element in the 2nd row, 1st column.

Given $B = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$ we can form the **augmented matrix**.

$\left(\begin{array}{cc|c} 2 & 4 & 5 \\ -1 & 3 & -2 \end{array}\right)$ which now 'includes' the matrix $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

We can use matrices to solve linear equations in the following way:
Consider the system of equations: $x - 2y + 3z = 14$
$2x + 3y - 4z = -16$
$3x - y - 2z = -1$

The matrix system is:
$$\begin{array}{cl} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 14 \\ 2 & 3 & -4 & -16 \\ 3 & -1 & -2 & -1 \end{array}\right) & \begin{array}{l} \text{- Row 1} \\ \text{- Row 2} \\ \text{- Row 3} \end{array} \end{array}$$

By performing **elementary row operations** (similar to simultaneous equations) we can reduce the 3 x 3 matrix to **upper triangular form** –

$$\begin{array}{lcl} \begin{array}{l} \text{Row 1} \\ \text{Row 2 – 2 x Row 1} \\ \text{Row 3 – 3 x Row 1} \end{array} & \left(\begin{array}{ccc|c} 1 & -2 & 3 & 14 \\ 0 & 7 & -10 & -44 \\ 0 & 5 & -11 & -43 \end{array}\right) & \begin{array}{l} \text{- New row 1} \\ \text{- New row 2} \\ \text{- New row 3} \end{array} \end{array}$$

$$\begin{array}{lcl} \begin{array}{l} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} - {}^{5}/_{7} \text{ x Row2} \end{array} & \left(\begin{array}{ccc|c} 1 & -2 & 3 & 14 \\ 0 & 7 & -10 & -44 \\ 0 & 0 & -{}^{27}/_{7} & -{}^{81}/_{7} \end{array}\right) & \begin{array}{l} \text{- New row 1} \\ \text{- New row 2} \\ \text{- New row 3} \end{array} \end{array}$$

This gives us the required upper triangular form with row 3:

$$-\frac{27}{7}z = -\frac{81}{7} \quad \Rightarrow \text{-}27z = \text{-}81 \quad \Rightarrow z = 3$$

From Row 2:
$7y - 10z = -44$
$7y - 30 = -44$
$7y = -14$
$y = -2$

From Row 1:
$x - 2y + 3z = 14$
$x + 4 + 9 = 14$
$x = 1$

Note: The elementary row operations must lead to a lower triangle of zero entries.

This method for the solution of a system of linear equations is called **Gaussian elimination**.

The system of equations can be written in matrix form as: $A\,X = B$.

In the example above $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ 3 & -1 & -2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 14 \\ -16 \\ -1 \end{pmatrix}$.

In some cases there will be no solution to the system of equations, for example
$x + 2y + z = 6$
$2x + y - z = 5$
$-2x - y + z = 0$

When Gaussian elimination is applied to this system, it results in $0z = 5$, i.e. no solution. This is referred to as **inconsistency**.

Another case is where there is an infinite family of solutions:

$$\begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5a \\ -4a \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 5a \\ 0 & -7 & -14a \end{array}\right) \quad \begin{array}{l} \text{R1} \\ \text{R2} - 2 \text{ x R1} \end{array}$$

$\Rightarrow -7y = -14a, \Rightarrow y = 2a$ and $x = a$, i.e. there is no unique solution.

The example in the Comments provides an example of ill-conditioning.

Exercises

1. Hunter	Page 150 Ex. 10.3, Q9, 10
2. Sadler & Thorning	Page 445 Ex 17F Q 1,2,3,5