Mathematics
Additional Question Bank
Mechanics Units 1 and 2 (AH)

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HIGHER STILL

Mathematics

Additional Question Bank
Mechanics Units 1 and 2
Advanced Higher

Support Materials

MATHEMATICS
Additional Question Bank
Advanced Higher Mechanics Units 1 and 2

1. INTRODUCTION
1.1 Background
The National Course in Advanced Higher Mathematics consists of two mandatory units, Mathematics 1(AH) and Mathematics 2(AH), followed by one of four optional units Mathematics 3(AH), Statistics 1(AH), Mechanics 1(AH) and Numerical Analysis 1(AH).

The National Course in Advanced Higher Applied Mathematics consists of a core of two units from Statistics 1(AH) and 2(AH), Mechanics 1(AH) and 2(AH) and Numerical Analysis 1(AH) and 2(AH), followed by one of four optional units Mathematics 1(AH), Statistics 1(AH), Mechanics 1(AH) and Numerical Analysis 1(AH). The optional unit chosen must not be a repeat of a unit taken by the candidate elsewhere in a Mathematics (AH) or Applied Mathematics (AH) course.

This pack contains a bank of additional questions for the two Mechanics Units, Mechanics 1(AH) and Mechanics 2(AH). Three other banks provide questions for Statistics 1 and 2(AH), Numerical Analysis 1 and 2(AH) and Mathematics 1, 2 and 3(AH).

For the most part, the content of the nine units available at Advanced Higher correlates highly with the content of the five Mathematics papers which were offered for the Certificate of Sixth Year Studies. The two main exceptions are in Statistics 2(AH) where most of the content is new and in Mathematics 2 and 3(AH) where some content from CSYS prior to 1992 is now reinstated.

The source of most of the questions in the banks is past CSYS Mathematics examination papers. Although these questions are in the public domain, they have the significant benefit of having undergone the question paper moderation procedures of the Scottish Qualifications Authority (SQA), (formerly the Scottish Examination Board), and have been scrutinised for clarity of language and mathematical accuracy. In addition, the difficulty levels attached to the questions are based on actual examination performance by candidates and the experience of examiners.

1.2 Structure and purpose
The structure of the banks is such that questions from future National Qualifications examinations for Advanced Higher Mathematics, Advanced Higher Applied Mathematics and from other sources available to users can be categorised similarly and added to the banks.

The purpose of the banks is to prepare students for course assessment and to generate evidence of attainment beyond the minimum competence necessary to pass the unit assessments for the component units of the chosen course. Centres are required to submit estimates of the bands candidates are likely to attain in the external course assessment and to retain the evidence of attainment on which estimates are based for use in the event of appeals. Using questions from the banks to obtain an assessment of the candidate's own unaided work should provide quality evidence of an estimate band. Centres may, of course, prefer to devise their own assessment materials, in which case modifying questions from the banks or creating new questions based on contexts used in questions in the banks may be helpful.

1.3 Quality of evidence

For assessment evidence in the form of prelim examinations or any other form of evidence to be fit for the purposes of estimates and appeals it is important that it covers as much of the course as possible. In Mathematics, evidence will normally be produced under supervision to ensure that it is the candidate's own unaided work.

1.4 Bank codes

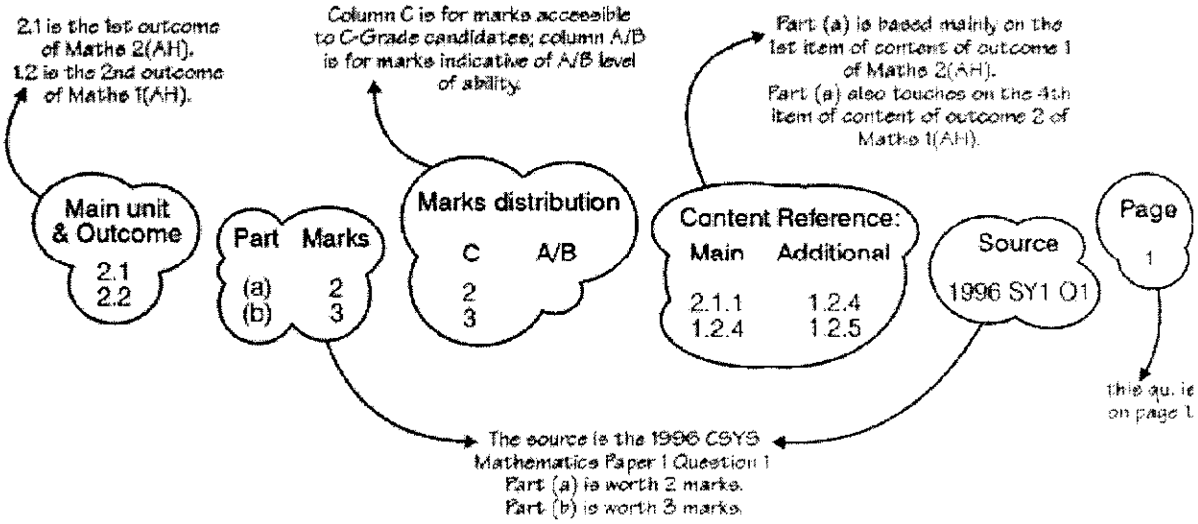
In the following sections of this Additional Question Bank, codes are used for ease of reference.

- Mechanics 1(AH) and Mechanics 2(AH) are referred to as Unit 1 and Unit 2 respectively.
- A 3-figure code has been applied to the items of course content as listed in the National Course Specifications for Advanced Higher Mathematics and Advanced Higher Applied Mathematics. For example, 2.1.10 is the reference to the tenth item of content in the first outcome of unit 2.
- A code 0.1 has been used to classify content which falls into the category of course grade descriptions.

Section 2 contains the full list of coded content for Mechanics 1 and 2(AH) in an abbreviated form. The document, SQA Applied Mathematics Advanced Higher:National Course Specification should be consulted for a full statement of course content and comment and course grade descriptions.

1.5 Additional questions

Section 3 of the Bank contains an analysis of the questions in grid form. The headings and abbreviations are explained below.



Section 4 lists each of the questions with, as a guide to marking, a simplified version of the actual marking instructions used in the examinations. Only one method of marking is illustrated and it should be noted that, in many instances, alternative methods are equally valid.

1.6 Important limitation on use of the Bank

Since national past examination papers are in the public domain, it is important, for reliability, that internal course assessments are constructed with questions from a wide spread of years.

Content List – Mechanics 1 (AH)

1.1 Motion in a straight line

1.1.1 know the meaning of position, displacement, velocity, acceleration, uniform speed, uniform acceleration, scalar quantity, vector quantity

Concepts of position, velocity and acceleration should be introduced using vectors.

Candidates should be very aware of the distinction between scalar and vector quantities, particularly in the case of speed and velocity.

1.1.2 draw, interpret and use distance/time, velocity/time and acceleration/time graphs

Candidates should be able to draw these graphs from numerical or graphical data.

1.1.3 know that the area under a velocity/time graph represents the distance travelled

1.1.4 know the rates of change $v = \frac{dx}{dt} = \dot{x}$ and $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \dot{v} = \ddot{x}$

Candidates should be familiar with the dot notation for differentiation with respect to time.

1.1.5 derive, by calculus methods, and use the equations governing motion in a straight line with constant acceleration, namely:

$v = u + at, s = ut + \frac{1}{2}at^2$ and from these,

$v^2 = u^2 + 2as, s = (u + v)t/2$

Candidates need to appreciate that these equations are for motion with *constant* acceleration only. The general technique is to use calculus.

1.1.6 solve analytically problems involving motion in one dimension under constant acceleration, including vertical motion under constant gravity

1.1.7 solve problems involving motion in one dimension where the acceleration is dependent on time, ie $a = \frac{dv}{dt} = f(t)$

1.2 Position, velocity and acceleration vectors including relative motion

1.2.1 know the meaning of the terms relative position, relative velocity and relative acceleration, air speed, ground speed and nearest approach

1.2.2 be familiar with the notation:

$\mathbf{r}_P$ for the position vector of P

$\mathbf{v}_P = \dot{\mathbf{r}}_P$ for the velocity vector of P

$\mathbf{a}_P = \dot{\mathbf{v}}_P = \ddot{\mathbf{r}}_P$ for the acceleration vector of P

$\overrightarrow{PQ} = \mathbf{r}_Q - \mathbf{r}_P$ for the position vector of Q relative to P

$\mathbf{v}_Q - \mathbf{v}_P = \dot{\mathbf{r}}_Q - \dot{\mathbf{r}}_P$ for the velocity of Q relative to P

$\mathbf{a}_P - \mathbf{a}_Q = \dot{\mathbf{v}}_P - \dot{\mathbf{v}}_Q = \ddot{\mathbf{r}}_P - \ddot{\mathbf{r}}_Q$ for the acceleration of Q relative to P

1.2.3 resolve vectors into components in two and three dimensions

This requires emphasis.

1.2.4 differentiate and integrate vector functions of time

1.2.5 use position, velocity and acceleration vectors and their components in two and three dimensions; these vectors may be functions of time

1.2.6 apply position, velocity and acceleration vectors to solve practical problems, including problems on the navigation of ships and aircraft and on the effect of winds and currents

Candidates should be able to solve such problems both by using trigonometric calculations in triangles and by vector components.

Solutions by scale drawing would not be accepted

1.2.7 solve problems involving collision courses and nearest approach

1.3 Motion of projectiles in a vertical plane

1.3.1 know the meaning of the terms projectile, velocity and angle of projection, trajectory, time of flight, range and constant gravity

Candidates also require to know how to resolve velocity into its horizontal and vertical components.

1.3.2 solve the vector equation $\ddot{\mathbf{r}} = -g\mathbf{j}$ to obtain $\mathbf{r}$ in terms of its horizontal and vertical components

The vector approach is particularly recommended.

1.3.3 obtain and solve the equations of motion $\ddot{x} = 0, \ddot{y} = -g$, obtaining expressions for $\dot{x}, \dot{y}, x$ and y in any particular case

1.3.4 find the time of flight, greatest height reached and range of a projectile

Only range on the horizontal plane through the point of projection is required.

1.3.5 find the maximum range of a projectile on a horizontal plane and the angle of projection to achieve this

1.3.6 find, and use, the equation of the trajectory of a projectile

Candidates should appreciate that this trajectory is a parabola.

1.3.9 solve problems in two-dimensional motion involving projectiles under a constant gravitational force and neglecting air resistance

Applications from ballistics and sport may be included and vector approaches should be used where appropriate.

1.4 Force and Newton's laws of motion

1.4.1 understand the terms mass, force, weight, momentum, balanced and unbalanced forces, resultant force, equilibrium, resistive forces

1.4.2 know Newton's first and third laws of motion

1.4.3 resolve forces in two dimensions to find their components

Resolution of velocities, etc. has been covered in previous sections.

1.4.4 combine forces to find resultant force

1.4.5 understand the concept of static and dynamic friction and limiting friction

1.4.6 understand the terms frictional force, normal reaction, coefficient of friction μ, angle of friction λ, and know the equations $F = \mu R$ and $\mu = \tan \lambda$

Balanced, unbalanced forces and equilibrium could arise here.

Candidates should understand that for stationary bodies, $F \leqslant \mu R$.

1.4.7 solve problems involving a particle or body in equilibrium under the action of certain forces

Forces could include weight, normal reaction, friction, tension in an inelastic string, etc.

1.4.8 know Newton's second law of motion, that force is the rate of change of momentum, and derive the equation $\mathbf{F} = m\mathbf{a}$

1.4.9 use this equation to form equations of motion to model practical problems on motion in a straight line

1.4.10 solve such equations modelling motion in one dimension, including cases where the acceleration is dependent on time

1.4.11 solve problems involving friction and problems on inclined planes

Both rough and smooth planes are required.

Mechanics 2 (AH)

2.1 Motion in a horizontal circle with uniform angular velocity

2.1.1 know the meaning of the terms angular velocity and angular acceleration

2.1.2 know that for motion in a circle of radius r, the radial and tangential components of velocity are 0 and $r\dot{\theta}\mathbf{e}_\theta$ respectively, and of acceleration are $-r\dot{\theta}^2\mathbf{e}_r$ and $r\ddot{\theta}\mathbf{e}_\theta$ respectively, where $\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$ are the unit vectors in the radial and tangential directions, respectively

Vectors should be used to establish these, starting from $\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$, where r is constant and θ is varying.

2.1.3 know the particular case where $\theta = \omega t$, ω being constant, when the equations are $\mathbf{r} = r\cos(\omega t)\mathbf{i} + r\sin(\omega t)\mathbf{j}$;
$\mathbf{v} = -r\omega\sin(\omega t)\mathbf{i} + r\omega\cos(\omega t)\mathbf{j}$;
$\mathbf{a} = -r\omega^2\cos(\omega t)\mathbf{i} - r\omega^2\sin(\omega t)\mathbf{j}$;
from which
$v = r\omega = r\dot{\theta}$; $a = r\omega^2 = r\dot{\theta}^2 = v^2/r$ and $\mathbf{a} = -\omega^2\mathbf{r}$

2.1.4 apply these equations to motion in a horizontal circle with uniform angular velocity including skidding and banking and other applications

Examples should include motion of cars round circular bends, with skidding and banking, the 'wall of death', the conical pendulum, etc.

2.1.5 know Newton's inverse square law of gravitation, namely that the magnitude of the gravitational force of attraction between two particles is inversely proportional to the square of the distance between the two particles

2.1.6 apply this to simplified examples of motion of satellites and moons

Circular orbits only.

2.1.7 find the time for one orbit, height above surface, etc

2.2 Simple harmonic motion

2.2.1 know the definition of simple harmonic motion (SHM) and the meaning of the terms oscillation, centre of oscillation, period, amplitude, frequency

2.2.2 know that SHM can be modelled by the equation $\ddot{x} = -\omega^2 x$

2.2.3 know the solutions $x = a\sin(\omega t + \alpha)$ and the special cases $x = a\sin(\omega t)$ and $x = a\cos(\omega t)$, of the SHM equation

At this stage these solutions can be verified or established from $\mathbf{r} = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$ rotating round a circle. Solution of second order differential equations is not required.

2.2.4 know and be able to verify that $v^2 = \omega^2(a^2 - x^2)$, where $v = \dot{x}$; $T = 2\pi/\omega$; maximum speed is ωa, the magnitude of the maximum acceleration is $\omega^2 a$ and when and where these arise

Proof using differential equations is not required here but will arise in the section of work on motion in a straight line later in this unit.

2.2.5 know the meaning of the term tension in the context of elastic strings and springs

2.2.6 know Hooke's law, the meaning of the terms natural length, l, modulus of elasticity, λ, and stiffness constant, k, and the connection between them, $\lambda = kl$

2.2.7 know the equation of motion of an oscillating mass and the meaning of the term position of equilibrium

2.2.8 apply the above to the solution of problems involving SHM

These will include problems involving elastic strings and springs, and small amplitude oscillations of a simple pendulum but not the compound pendulum.

2.3 Principles of momentum and impulse

2.3.1 know that force is the rate of change of momentum

This was introduced in Mechanics 1 (AH).

2.3.2 know that impulse is change in momentum i.e.
$\mathbf{I} = m\mathbf{v} - m\mathbf{u} = \int \mathbf{F}dt$

2.3.3 understand the concept of conservation of linear momentum

2.3.4 solve problems on linear motion such as motion in lifts, recoil of a gun, pile-drivers, etc.

The equation $\mathbf{F} = m\mathbf{a}$ is again involved here. Equations of motion with constant acceleration could recur.

2.4 Principles of work, power and energy

2.4.1 know the meaning of the terms work, power, potential energy, kinetic energy

2.4.2 understand the concept of work

Candidates should appreciate that work can be done by or against a force.

2.4.3 calculate the work done by a constant force in one and two dimensions, ie, $W = Fd$ (one dimension); $W = \mathbf{F.d}$ (two dimensions)

2.4.4 calculate the work done in rectilinear motion by a variable force using integration, i.e. $W = \int \mathbf{F.i}\,dx$; $W = \int \mathbf{F.v}\,dt$, where $\mathbf{v} = \frac{dx}{dt}\mathbf{i}$

2.4.5 understand the concept of power as the rate of doing work, i.e. $P = \frac{dW}{dt} = \mathbf{F.v}$ (constant force), and apply this in practical examples

Examples can be taken from transport, sport, fairgrounds, etc.

2.4.6 understand the concept of energy and the difference between kinetic (E_K) and potential (E_P) energy

2.4.7 know that $E_K = \frac{1}{2}mv^2$

2.4.8 know that the potential energy associated with:

a. a uniform gravitational field is $E_P = mgh$

b. Hooke's law is $E_P = \frac{1}{2}k(\text{extension})^2$

Link with simple harmonic motion.

c. Newton's inverse square law is $E_P = \frac{GMm}{r}$

Link with motion in a horizontal circle.

2.4.9 understand and apply the work-energy principle

2.4.10 understand the meaning of conservative forces like gravity, and non-conservative forces like friction

2.4.11 know and apply the energy equation $E_K + E_P = \text{constant}$, including the situation of motion in a vertical circle

2.5 Motion in a straight line, where the solution of first order differential equations is required

2.5.1 know that $a = v\frac{dv}{dx}$ as well as $\frac{dv}{dt}$

2.5.2 use Newton's law of motion, $\mathbf{F} = m\mathbf{a}$, to form first order differential equations to model practical problems, where the acceleration is dependent on displacement or velocity, i.e. $\frac{dv}{dt} = f(v)$; $v\frac{dv}{dx} = f(x)$; $v\frac{dv}{dx} = f(v)$.

2.5.3 solve such differential equations by the method of separation of variables

It may be necessary to teach this solution technique, depending on the mathematical background of the candidates.

Examples will be straightforward with integrals which are covered in Mathematics 1, 2 (AH). If more complex, then the anti-derivative will be given.

2.5.4 derive the equation $v^2 = \omega^2(a^2 - x^2)$ by solving $v\frac{dv}{dx} = -\omega^2 x$

2.5.5 know the meaning of the terms terminal velocity, escape velocity and resistance per unit mass and solve problems involving differential equations and incorporating any of these terms or making use of $F = \frac{P}{v}$

This section can involve knowledge and skills from other topics within this unit.

Main unit & Outcome	Part	Marks	Marks distribution C	A/B	Content Reference: Main	Additional	Source	Page
2.2	(a)	3	3		2.2.4	2.2.1	1996 SY5 Q1	1
2.2	(b)	2	2		2.2.3			
1.2	(a)	3	3		1.2.4/6	1.2.5	1996 SY5 Q2	2
1.2	(b)	5	5		1.2.7/4	1.2.5		
1.4		5	5		1.4.11/8	1.4.6/3	1996 SY5 Q3	3
2.4		4	2		2.4.3	1.4.3		
2.1		4	4		2.1.3/4	1.4.3	1996 SY5 Q4	4
2.1		1	1		2.1.4			
2.3		5	3		2.3.3		1996 SY5 Q5	5
1.2			2		1.2.5	1.1.1		
1.3		4	4		1.3.4		1996 SY5 Q6	6
1.3		6	1	5	1.3.9	1.3.5/4		
1.3		5		5	1.3.9	1.3.6		
2.5		8		8	2.5.2/3	2.5.1	1996 SY5 Q7	8
2.5		5		5	2.5.2/3	2.5.1		
2.4		2	2		2.4.8			
2.4		2	2		2.4.11	2.4.7/8	1996 SY5 Q8	10
2.4		6		6	2.4.11	2.1.3		
2.4		2	2		2.1.3			
2.4		5	1	4	2.4.11	2.1.3		
2.5/4		3	3		2.5.2/5	2.5.1	1996 SY5 Q9	12
2.5		7		7	2.5.3			
2.5		5		5	2.5.3	2.5.1		
2.2		2	2		2.2.6	1.4.1	1996 SY5 Q10	14

Main unit & Outcome	Part	Marks	Marks distribution C	A/B	Content Reference: Main	Additional	Source	Page
2.2		5	5		2.2.4	2.2.1	1997 SY5 Q1	15
1.1	(a)	5	4	1	1.1.5/6		1997 SY5 Q2	16
1.1	(b)	1	1		1.1.6			
2.1	(a)	3	3		2.1.5/6		1997 SY5 Q3	17
2.1	(b)	3	3		2.1.7	2.1.3		
1.2/4		6	4	2	1.2.4/2.4.4	1.1.7	1997 SY5 Q4	18
1.2		7	5	2	1.2.7/3	1.2.2	1997 SY5 Q5	19
1.3	(a)	5	5		1.3.6	1.3.1	1997 SY5 Q6	20
1.3	(b)	6		6	1.3.9/6			
1.3	(c)	4		4	1.3.9/6			
2.5	(a)	3	3		2.5.5	1.4.1/11	1997 SY5 Q7	23
2.5	(b)	2	2		2.5.2	1.4.11		
2.5	(c)	5		5	2.5.3	2.5.1		
2.5	(d)	5		5	2.5.3	2.5.1		
2.3	(a)	3	3		2.3.3		1997 SY5 Q8	26
2.4	(b)	8		8	2.4.11	2.4.7/8,2.1.3		
2.4	(c)	4	2	2	2.4.11	2.4.6/2.1.3		
1.4	(a)	3	3		1.4.3/6	1.4.5	1997 SY5 Q9	29
1.4	(b)	6		6	1.4.11/3	1.4.6		
1.4	(c)	6	2	4	1.4.11/3	1.4.6		
2.2	(a)	2	2		2.2.6	2.2.5	1997 SY5 Q10	31
2.2	(b)	3		3	2.2.7	1.4.8/1.1.4		
2.2		3	3		2.2.2/3	2.2.1		

Main unit & Outcome	Part	Marks	Marks distribution C	A/B	Content Reference: Main	Additional	Source	Page
2.1		3	3		2.1.5/6	2.1.7/3	1998 SY5 Q1	33
2.4		2	2		2.4.5	1.4.1	1998 SY5 Q2	34
2.4		3	3		2.4.5	1.4.11		
1.2		6	6		1.2.4/7	1.2.2/6	1998 SY5 Q3	35
1.4		6	6		1.4.3/6	1.4.11/5	1998 SY5 Q4	36
2.3		3	2	1	2.3.2		1998 SY5 Q5	37
2.1		7	5	2	2.1.3/4	1.4.3	1998 SY5 Q6	38
1.3	(a)	3	3		1.3.4	1.3.3	1998 SY5 Q7	40
1.3	(b)	6		6	1.3.9	1.3.3		
1.3	(c)	3	1	2	1.3.9	1.3.4		
1.3	(d)	3	1	2	1.3.9	1.3.4		
2.5	(a)	2		2	2.5.2	2.5.1	1998 SY5 Q8	42
2.5		2	2		2.5.5			
2.5	(b)	7	1	6	2.5.3			
2.5	(c)	4	1	3	2.5.3	2.5.1		
2.4	(a)	5		5	2.4.11	2.4.6/3	1998 SY5 Q9	44
2.4	(b)	2	2		2.4.11			
2.4	(c)(i)	3	3		2.4.11			
2.4	(c)(ii)	5		5	2.4.11	1.3.4		
2.2	(a)	2	2		2.2.8		1998 SY5 Q10	47
2.2	(b)	7		7	2.2.6/8			
2.4	(c)	3		3	2.4.8			
2.4	(d)	3	2	1	2.4.8			
2.5	(a)	3		3	2.5.4	2.5.1	1998 SY5 Q11	49
2.2	(b)	6	4	2	2.5.4	2.5.1,2.2.8		

Main unit & Outcome	Part	Marks	Marks distribution C	A/B	Content Reference: Main	Additional	Source	Page
1.1,1.2		5	5		1.1.7	1.1.4,1.2.4	1999 SY5 Q1	50
2.2		6	1	5	2.2.3/4	2.2.1	1999 SY5 Q2	51
1.2		4	4		1.2.4/7	1.2.1	1999 SY5 Q3	52
1.4		6	6		1.4.3/4	1.4.11	1999 SY5 Q4	53
2.3	(a)	3	3		2.3.3	2.3.4	1999 SY5 Q5	54
2.3	(b)	2		2	2.3.4	1.2.5		
2.1		4	4		2.1.3/4	1.4.6	1999 SY5 Q6	55
1.3	(a)	4	4		1.3.6	1.3.1	1999 SY5 Q7	56
	(b)	4		4	1.3.6/9			
	(c)(i)	5		5	1.3.4			
	(c)(ii)	2	1	1	1.3.4			
2.4	(a)	4	2	2	2.4.5	1.4.1	1999 SY5 Q8	58
2.5	(b)	6	1	5	2.5.2/3	2.5.5		
2.5	(c)	5	1	4	2.5.2/3			
2.4	(a)	6		6	2.4.11	2.4.6/2.1.3	1999 SY5 Q9	60
2.1	(b)(i)	3	2	1	2.1.3			
2.4	(b)(ii)	3	2	1	2.4.11			
2.4	(c)	3		3	2.4.11	2.4.7/8		
2.1	(a)(i)	3	3		2.1.5	2.5.1	1999 SY5 Q10	62
2.5	(a)(ii)	7		7	2.5.3	2.5.5		
2.1	(b)	5	2	3	2.1.6/5	2.1.3/7		

Main unit & Outcome	Part	Marks	Marks distribution C	A/B	Content Reference: Main	Additional	Source	Page
1.4,1.1		4	4		1.4.6	1.4.5	2000 SY5 Q1	64
					1.4.8	1.4.9		
					1.1.5			
2.2		3	3		2.2.4	2.2.1	2000 SY5 Q2	65
2.2		2	2		2.2.4			
2.1		5	5		2.1.6/5	2.1.3/7	2000 SY5 Q3	66
2.1		4	4		2.1.4	2.1.3	2000 SY5 Q4	67
						1.4.3		
2.4		3	3		2.4.3		2000 SY5 Q5	68
1.2	(a)	4	4		1.2.4/2	1.2.5/1	2000 SY5 Q6	69
1.2	(b)	2	2		1.2.4/2	1.2.5/1		
1.2	(c)	3		3	1.2.7	1.2.5		
1.3	(a)	6		6	1.3.3/6	1.3.1	2000 SY5 Q7	70
1.3	(b)(i)	4	4		1.3.4			
1.3	(ii)	5		5	1.3.9	1.3.4		
2.4	(a)(i)	6		6	2.4.11	2.4.6/10,2.1.3	2000 SY5 Q8	72
2.4	(a)(ii)	2	2					
2.4	(b)	5	1	4	2.4.11	2.4.6/10,2.1.3		
2.4	(c)	2	2					
2.5	(a)	8	3	5	2.5.2/3	2.5.5	2000 SY5 Q9	75
2.5	(b)	7	1	6	2.5.2/3			
1.4	(a)	4	4		1.4.6	1.4.3	2000 SY5 Q10	77
1.4	(b)(i)	3		3	1.4.6	1.4.3		
1.4	(b)(ii)	8	1	7	1.4.11	1.4.5		

A particle performs simple harmonic motion in a straight line between points A and B with period 0·6 seconds. Initially it was projected from C, the midpoint of AB, with speed $\frac{\pi}{5}$ m s^{-1} towards B.

Calculate

(a) the length of AB, and **3**

(b) the time taken by the particle to move directly from C to D, the midpoint of CB. **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	2.2	3		2.2.4	2.2.1	1996 SY5 Q1
(b)	2	2.2	2		2.2.3		

(a) Max. velocity $\omega a = \frac{\pi}{5}$. **1**

Period, $\frac{2\pi}{\omega} = 0{\cdot}6 \Rightarrow \omega = \frac{2\pi}{0{\cdot}6} = \frac{10\pi}{3}$ **1**

$$a = \frac{\pi}{5} \times \frac{3}{10\pi} = 0{\cdot}06 \text{ m}$$

$$AB = 0{\cdot}12 \text{ m}$$ **1**

(b)

$$0{\cdot}03 = 0{\cdot}06 \sin \frac{10\pi}{3}t$$ **1**

$$\Rightarrow \sin \frac{10\pi}{3}t = 0{\cdot}5$$

$$\Rightarrow \frac{10\pi}{3}t = \frac{\pi}{6}$$

$$\Rightarrow t = 0{\cdot}05 \text{ seconds}$$ **1**

A particle P has position vector $\mathbf{r}_P = (t+1)^2\mathbf{i} - 3(t-3)\mathbf{j}$, where t is the time in seconds and $\mathbf{i}$, $\mathbf{j}$ are the unit vectors in the directions of rectangular axes Ox and Oy respectively. A second particle Q has acceleration $4\mathbf{j}$. At time $t = 0$, the particle Q has velocity $6\mathbf{i} + 4\mathbf{j}$ and position vector $-2\mathbf{i} - 30\mathbf{j}$.

(a) Show that the particles are always moving at right angles to each other. **3**

(b) Show that the particles collide, and calculate the time and position at which the collision occurs. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	1.2	3		1.2.4/6	1.2.5	1996 SY5 Q2
(b)	5	1.2	5		1.2.7/4	1.2.5	

(a)

$$\mathbf{r}_P = (t+1)^2\mathbf{i} - 3(t-3)\mathbf{j}$$

$$\dot{\mathbf{r}}_P = 2(t+1)\mathbf{i} - 3\mathbf{j} \qquad \mathbf{1}$$

$$\ddot{\mathbf{r}}_Q = 4\mathbf{j}$$

$$\dot{\mathbf{r}}_Q = 6\mathbf{i} + 4(t+1)\mathbf{j} \qquad \mathbf{1}$$

$$\dot{\mathbf{r}}_P.\dot{\mathbf{r}}_Q = 2(t+1)\times 6 - 3\times 4(t+1) = 0$$

$$\Rightarrow \dot{\mathbf{r}}_P \text{ is perpendicular to } \dot{\mathbf{r}}_Q \qquad \mathbf{1}$$

(b)

$$\mathbf{r}_Q = (6t-2)\mathbf{i} + \left(2t^2 + 4t - 30\right)\mathbf{j} \qquad \mathbf{1}$$

Equating coefficients

$$(t+1)^2 = 6t - 2 \qquad (A)$$

$$-3(t-3) = 2t^2 + 4t - 30 \qquad (B) \qquad \mathbf{1}$$

Solving quadratics

From A, $t = 1$ or 3

From B, $t = 3$ or $-\frac{13}{2}$ **1**

$$t = 3 \qquad \mathbf{1}$$

$$\mathbf{r}_P = \mathbf{r}_Q = 16\mathbf{i} \qquad \mathbf{1}$$

A block of mass m kilograms is projected with speed v m s^{-1} up a line of greatest slope of a rough plane inclined at an angle α to the horizontal. The coefficient of friction between the block and the surface of the plane is μ. The block travels a distance s metres up the plane before coming to rest.

Find an expression for s in terms of v, α, μ and g, where g is the magnitude of the acceleration due to gravity. **5**

Show that the work done against friction is

$$\frac{\mu m v^2}{2(\mu + \tan\alpha)}.$$ **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	1.4	5		1.4.11/8	1.4.6/3	1996 SY5 Q3
	4	2.4	2		2.4.3	1.4.3	

$$N - mg\cos\alpha = 0$$ **1**

$$ma = -F - mg\sin\alpha$$ **1**

$$ma = -\mu mg\cos\alpha - mg\sin\alpha$$

$$a = -g(\mu\cos\alpha + \sin\alpha)$$ **1**

$$0 = v^2 - 2g(\mu\cos\alpha + \sin\alpha)s$$ **1**

$$s = \frac{v^2}{2g(\mu\cos\alpha + \sin\alpha)}$$ **1**

$$\text{Work done} = \mu mg\cos\alpha \times \frac{v^2}{2g(\mu\cos\alpha + \sin\alpha)}$$ **1**

$$= \frac{\mu m v^2}{2(\mu + \tan\alpha)}$$ **1**

A particle of mass m is attached by a light inextensible string to a fixed point O. The particle moves in a horizontal circle whose centre is a distance h vertically below O. Derive an expression for h in terms of ω and g, where ω is the angular speed of the particle and g is the magnitude of the acceleration due to gravity. **4**

What is the effect on h of doubling the angular speed? **1**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	4	2.1	4		2.1.3/4	1.4.3	1996 SY5 Q4
	1	2.1	1		2.1.4		

$$T\cos\theta - mg = 0$$ **1**

$$r = h\tan\theta$$ **1**

$$m\omega^2 h\tan\theta = T\sin\theta$$ **1**

$$\frac{m\omega^2 h\tan\theta}{mg} = \frac{T\sin\theta}{T\cos\theta}$$

$$h = \frac{g}{\omega^2}$$ **1**

$$\omega' = 2\omega$$

$$h' = \frac{g}{(2\omega)^2} = \frac{g}{4\omega^2}$$

$$h' = \frac{1}{4}h$$ **1**

In vehicle safety trials, a car of mass 1200 kilograms and a van of mass 2000 kilograms were made to crash into each other on a skid-pan. The two vehicles locked together and moved on as one combined mass.

Before impact, taking the x-axis as the direction of the car, the velocity of the car was $18\mathbf{i}$ m s^{-1} and the velocity of the van was $(5\mathbf{i} + 12\mathbf{j})$ m s^{-1}, where **i**, **j** are the unit vectors in the directions of the rectangular axes Ox and Oy respectively.

Given that friction can be ignored, calculate the speed and direction of the combined mass immediately after the crash. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	2.3	3		2.3.3		1996 SY5 Q5
		1.2	2		1.2.5	1.1.1	

$1200 \times 18\mathbf{i} + 2000(5\mathbf{i} + 12\mathbf{j}) = 3200\mathbf{v}$ **1 for correct equation**
1 for substitution

$$\mathbf{v} = \frac{1}{8}(79\mathbf{i} + 60\mathbf{j})$$ **1**

$$\text{Speed} = \frac{1}{8}\sqrt{79^2 + 60^2} \approx 12{\cdot}4 \text{ m s}^{-1}$$ **1**

Direction is at an angle $\tan^{-1}\left(\frac{60}{79}\right) \approx 37{\cdot}2°$ with Ox. **1**

A particle is projected with speed V at an angle α above the horizontal. Derive an expression for the range of the particle on a horizontal plane in terms of V, α and g, where g is the magnitude of the acceleration due to gravity. **4**

A cricketer can throw a ball a **maximum** range of 60 metres to the wicket-keeper. Given that the ball is released from a point 1·5 metres above the ground and caught at the same height, calculate the speed with which it is thrown and the maximum height reached above ground level. (Air resistance may be ignored.) **6**

How much further horizontally will the ball travel before striking the ground if the wicket-keeper fails to make contact with the ball? **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	4	1.3	4		1.3.4		1996 SY5 Q6
	6	1.3	1	5	1.3.9	1.3.5/4	
	5	1.3		5	1.3.9	1.3.6	

$$r = V\cos\alpha\ t \qquad \mathbf{1}$$

$$0 = V\sin\alpha\ t - \frac{1}{2}gt^2 \qquad \mathbf{1}$$

$$t = \frac{2V\sin\alpha}{g} \qquad \mathbf{1}$$

$$r = V\cos\alpha\, t \times \frac{2V\sin\alpha}{g}$$

$$= \frac{V^2\sin 2\alpha}{g} \qquad \mathbf{1}$$

Max. range, $\frac{V^2}{g} = 60$ **1**

$$V = 14\sqrt{3} \approx 24{\cdot}2\ \text{m s}^{-1} \qquad \mathbf{1}$$

Either		Or	
$0 = (V\sin 45°)^2 - 2gh$	**2**	$0 = V\sin 45° - gt$	**1**
$h = \frac{30g}{2g}$		$t = 1{\cdot}75$ sec	**1**
		$h = 24{\cdot}2 \times 1{\cdot}75 - 0{\cdot}5 \times 9{\cdot}8 \times 1{\cdot}75^2$	
$= 15$ m	**1**	$= 15$ (or $14{\cdot}9$) m	**1**

Max. height = 16·5 (16·4) metres **1**

$$-1{\cdot}5 = 24.2\sin 45°\, t - \frac{1}{2} \times 9.8t^2 \qquad \mathbf{1}$$

$$4{\cdot}9t^2 - 17{\cdot}15t - 1.5 = 0 \qquad \mathbf{1}$$

$$t = 3{\cdot}59 \text{ seconds} \qquad \mathbf{1}$$

$$\text{Distance} = 17{\cdot}15 \times 3{\cdot}59 = 61{\cdot}5 \text{ metres} \qquad \mathbf{1}$$

The ball travels 1·5 metres further. **1**

A body of mass m falls from rest to the ground under constant gravity from a height h. It experiences a resistange of kv^2 per unit mass, where k is a positive constant and v is the speed at time t. Show that the speed V with which the body strikes the ground is given by the expression

$$V = \sqrt{\frac{g}{k}(1 - e^{-2kh})},$$

where g is the magnitude of the acceleration due to gravity. **8**

Given that the body rebounds from the ground with the same speed V with which it struck the ground, and that it experiences the same resistance as before, find an expression in terms of V, k and g for the height to which the body rises. **5**

Show that the energy lost by the body in falling once and rising once is equal to

$$\frac{mg}{2k} \ln\left|\frac{g^2}{g^2 - k^2V^4}\right|.$$ **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	8	2.5		8	2.5.2/3	2.5.1	1996 SY5 Q7
	5	2.5		5	2.5.2/3	2.5.1	
	2	2.4	2		2.4.8		

$$mv\frac{dv}{dx} = mg - mkv^2 \qquad \textbf{2}$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\int_0^h dx = \int_0^v \frac{v\,dv}{g - kv^2} \qquad \textbf{1 for limits}$$

1 for rearrangement

$$h = \left[-\frac{1}{2k}\ln\left(g - kV^2\right)\right]_0^V \qquad \textbf{1}$$

$$= -\frac{1}{2k}\ln\frac{g - kV^2}{g} \qquad \textbf{1}$$

$$\ln\frac{g - kV^2}{g} = -2kh$$

$$g - kV^2 = ge^{-2kh} \qquad \textbf{1}$$

$$kV^2 = g\left(1 - e^{-2kh}\right)$$

$$V = \sqrt{\frac{g}{k}\left(1 - e^{-2kh}\right)} \qquad \textbf{1}$$

$$mv\frac{dv}{dx} = -mg - mkv^2 \qquad \textbf{1}$$

$$v\frac{dv}{dx} = -\left(g + kv^2\right)$$

$$\int_0^{h'} dx = -\int_V^0 \frac{v\,dv}{g + kv^2} \qquad \textbf{2}$$

$$h' = \frac{1}{2k}\left[\ln\left(g + kv^2\right)\right]_0^V \qquad \textbf{1}$$

$$= \frac{1}{2k}\ln\left(\frac{g + kV^2}{g}\right) \qquad \textbf{1}$$

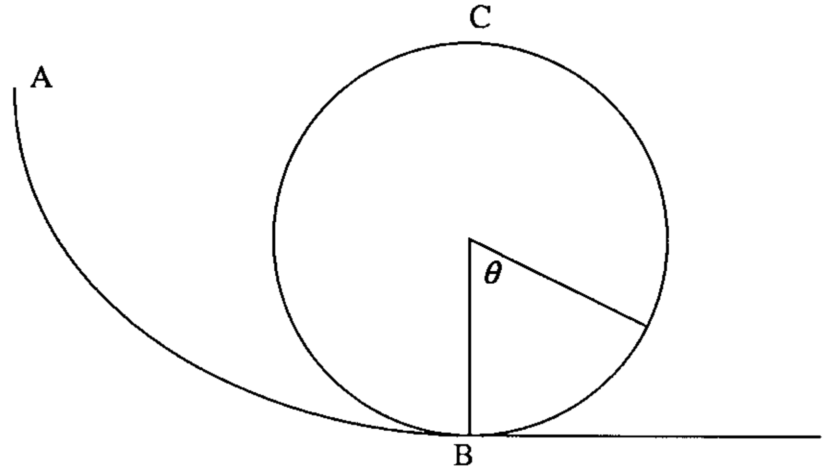
$$\text{Loss in energy} = mgh - mgh'$$

$$= mg\left\{\frac{1}{2k}\ln\frac{g}{g - kV^2} - \frac{1}{2k}\ln\frac{g + kV^2}{g}\right\} \qquad \textbf{1}$$

$$= \frac{mg}{2k}\ln\left\{\frac{g}{g - kV^2} \times \frac{g}{g + kV^2}\right\}$$

$$= \frac{mg}{2k}\ln\left\{\frac{g^2}{g^2 - k^2V^4}\right\} \qquad \textbf{1}$$

A toy car runs on a track, which is ;assumed to be smooth and to lie in a vertical plane. The track slopes down from a point A at height h above the floor. The track reaches the floor at point B, after which it enters a circular loop of radius r with BC as its vertical diameter, before levelling out after passing B again.



A car of mass m is released from rest at A. Find an expression for its speed at B. **2**

Given that $h = 2r$, find an expression in terms of m, g and θ, for the normal reaction between the track and the car, where θ is the angle between the radius to the car and the downward vertical and g is the magnitude of the acceleration due to gravity. **6**

Hence find the height above the floor of the point at which the car loses contact with the track. **2**

The height of the point A is now adjusted to enable the car to complete the vertical circle on release from rest at A. Find the least height by which A must be raised, and the corresponding speed at C. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	2	2.4	2		2.4.11	2.4.7/8	1996 SY5 Q8
	6	2.4		6	2.4.11	2.1.3	
	2	2.4	2		2.1.3		
	5	2.4	1	4	2.4.11	2.1.3	

$$\frac{1}{2}mu^2 = mgh \qquad \mathbf{1}$$

$$u^2 = 2gh \qquad \mathbf{1}$$

$$\frac{1}{2}mv^2 + mgr(1 - \cos\theta) = \frac{1}{2}mu^2 \qquad \mathbf{2}$$

$$v^2 = 4gr - 2gr(1 - \cos\theta)$$

$$= 2gr(1 + \cos\theta) \qquad \mathbf{1}$$

$$\frac{mv^2}{r} = R - mg\cos\theta \qquad \mathbf{2}$$

$$R = 2mg(1 + \cos\theta) + mg\cos\theta$$

$$= mg(2 + 3\cos\theta) \qquad \mathbf{1}$$

When $R = 0, \cos\theta = -\frac{2}{3}$ **1**

height = $r + \frac{2}{3}r = \frac{5}{3}r$ **1**

$$\text{At C, } \frac{mv^2}{r} = mg$$

$$v^2 = gr \qquad \mathbf{1}$$

V = speed at B

$$\frac{1}{2}mV^2 = mg \times 2r + \frac{1}{2}mv^2$$

$$V^2 = 5gr \qquad \mathbf{1}$$

$$\text{At A, } \frac{1}{2}mV^2 = mgh' \qquad \mathbf{1}$$

$$h' = \frac{5}{2}r \qquad \mathbf{1}$$

The height is increased by $\frac{1}{2}r$. **1**

A locomotive, having mass 100 tonnes and travelling on a horizontal track, works at a constant power of 1600kW. It experiences a resistance of magnitude 1000v newtons, where v is the speed of the locomotive in metres per second. Show that a differential equation for the motion is

$$\frac{dv}{dt} = \frac{1600 - v^2}{100v}.$$ **3**

Given that the locomotive starts from rest, derive an expression for v in terms of t. **7**

Given that $\frac{v^2}{1600 - v^2} = \frac{20}{40 - v} + \frac{20}{40 + v} - 1$, find an expression in terms of v for the distance travelled in reaching that speed. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	2.5/4	3		2.5.2/5	2.5.1	1996 SY5 Q9
	7	2.5		7	2.5.3		
	5	2.5		5	2.5.3	2.5.1	

$$m\frac{dv}{dt} = F - R \qquad \textbf{1}$$

$$= \frac{P}{v} - 1000v \qquad \textbf{1}$$

$$10^5\frac{dv}{dt} = \frac{1{\cdot}6 \times 10^6}{v} - 10^3 v$$

$$\frac{dv}{dt} = \frac{1600 - v^2}{100v} \qquad \textbf{1}$$

$$\int_0^t dt = \int_0^v \frac{100v\,dv}{1600 - v^2}$$ **1 for limits**

1 for manipulation

$$t = \left[-50\ln\left(1600 - v^2\right)\right]_0^v$$ **1 for technique of integration**

1 for evaluation

$$= 50\ln\left(\frac{1600}{1600 - v^2}\right) \qquad \textbf{1}$$

$$1600 - v^2 = 1600e^{-t/50} \qquad \textbf{1}$$

$$v^2 = 1600\left(1 - e^{-t/50}\right)$$

$$v = 40\sqrt{1 - e^{-t/50}} \qquad \textbf{1}$$

$$v\frac{dv}{dx} = \frac{1600 - v^2}{100v} \qquad \textbf{1}$$

$$\int_0^x dx = \int_0^v \frac{100v^2}{1600 - v^2}\,dv \qquad \textbf{1}$$

$$x = 100\int_0^v \left\{\frac{20}{40 - v} + \frac{20}{40 + v} - 1\right\}dv \qquad \textbf{1}$$

$$= 100\left[-20\ln(40 - v) + 20\ln(40 + v) - v\right]_0^v \qquad \textbf{1}$$

$$= 2000\ln\left(\frac{40 + v}{40 - v}\right) - 100v \qquad \textbf{1}$$

A particle of mass 0·025 kilograms is attached to one end of a light spring of natural length 0·25 metres and stiffness constant k. The other end of the spring is attached to a fixed point O and the particle hangs in equilibrium at the point E, which is 0·299 metres vertically below O. Find k. **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	2	2.2	2		2.2.6	1.4.1	1996 SY5 Q10

In equilibrium $k \times 0{\cdot}049 = 0{\cdot}025 \times 9{\cdot}8$ **1**

$k = 5$ newtons/metre **1**

The maximum speed of a particle executing simple harmonic motion is 13 centimetres per second. When the particle has moved 15 centimetres from its centre of oscillation, its speed is 12 centimetres per second.
Calculate the period and the amplitude of the motion. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	2.2	5		2.2.4	2.2.1	1997 SY5 Q1

Maximum speed = $a\omega \Rightarrow a\omega = 0{\cdot}13$ **1**

$$v^2 = \omega^2(a^2 - x^2)$$

$$v^2 = a^2\omega^2 - x^2\omega^2$$

$$x^2\omega^2 = a^2\omega^2 - v^2$$

$$\omega^2 = \frac{a^2\omega^2 - v^2}{x^2}$$

$$= \frac{0{\cdot}13^2 - 0{\cdot}12^2}{0{\cdot}15^2} = \frac{1}{9}$$ **1**

$$\Rightarrow \omega = \frac{1}{3}$$ **1**

$$\Rightarrow T = \frac{2\pi}{\omega} = 6\pi \text{ seconds}$$ **1**

$$a = \frac{0{\cdot}13}{\omega} = 0{\cdot}39 \text{ m} = 39 \text{ cm}$$ **1**

A fish is held out at the top of a ladder at a height of h metres above the water surface of a pool. A seal rises vertically from the water, directly below the fish, with an initial speed of u metres per second. The fish is dropped at exactly the same instant as the seal emerges from the water. Given that the seal is still rising when it catches the fish, show that:

(a) $u > \sqrt{gh}$, where g is the magnitude of the acceleration due to gravity; **5**

(b) the seal catches the fish at a height of

$$h\left(1 - \frac{gh}{2u^2}\right) \text{ metres}$$

above the water surface of the pool. **1**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	5	1.1	4	1	1.1.5/6		1997 SY5 Q2
(b)	1	1.1	1		1.1.6		

(a) f = position of fish; s = position of seal

$$\ddot{f} = -g \qquad \ddot{s} = -g$$

$$\dot{f} = -gt + c \qquad \dot{s} = -gt + u$$

$$\text{when } t = 0, \dot{f} = 0 \Rightarrow c = 0 \qquad s = -\tfrac{1}{2}gt^2 + ut + d$$

$$\dot{f} = -gt \qquad \text{when } t = 0, s = 0 \Rightarrow d = 0$$

$$f = -\tfrac{1}{2}gt^2 + h \quad \mathbf{1} \qquad s = -\tfrac{1}{2}gt^2 + ut \quad \mathbf{1}$$

Fish caught when $f = s$ $\quad -\tfrac{1}{2}gt^2 + h = -\tfrac{1}{2}gt^2 + ut$

$$\Rightarrow \quad t = \frac{h}{u} \qquad \mathbf{1}$$

$$\dot{s} = \text{velocity of seal}$$
$$= u - gt$$

For this to be positive

$$u > gt \qquad \mathbf{1}$$

$$\Rightarrow \quad u > \frac{gh}{u} \qquad t = \frac{h}{u}$$

$$\Rightarrow \quad u^2 > gh$$

$$\Rightarrow \quad u > \sqrt{gh} \qquad \mathbf{1}$$

(b) Substitute $t = \frac{h}{u}$ into f or s

$$f = -\frac{1}{2}gt^2 + h$$

$$= -\frac{1}{2}g\frac{h^2}{u^2} + h = h\left(1 - \frac{gh}{2u^2}\right) \qquad \mathbf{1}$$

A satellite moves in a circular orbit around a planet in the plane of the planet's equator and at a height of 600 kilometres above the surface of the planet. The magnitude of the acceleration due to gravity at the surface of the planet is $11{\cdot}2\ \text{m s}^{-2}$ and the radius of the planet is 8600 kilometres.

(a) Given that Newton's inverse square law of gravitation applies, calculate the magnitude of the gravitational acceleration experienced by the satellite. **3**

(b) Find also the time taken by the satellite to complete one orbit. **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	2.1	3		2.1.5/6		1997 SY5 Q3
(b)	3	2.1	3		2.1.7	2.1.3	

(a) At surface

$$mg_p = \frac{GMm}{R^2}$$

$$\Rightarrow \quad GM = g_pR^2 \qquad \mathbf{1}$$

where g_p is the gravitational field strength on surface of planet and R is the radius of the planet.

At satellite

$$mg_s = \frac{GMm}{r^2}$$

$$\Rightarrow \quad g_s = \frac{g_pR^2}{r^2} \qquad \mathbf{1}$$

$$= \frac{11{\cdot}2 \times 8600\,000^2}{9200\,000^2} = 9{\cdot}79\ \text{m s}^{-2} \qquad \mathbf{1}$$

(b)

$$mr\omega^2 = \frac{GMm}{r^2} \qquad \mathbf{1}$$

$$\omega^2 = \frac{g_pR^2}{r^3}$$

$$\left(\frac{2\pi}{T}\right)^2 = \frac{g_pR^2}{r^3} \qquad \mathbf{1}$$

$$\frac{4\pi^2}{T^2} = \frac{g_pR^2}{r^3}$$

$$T^2 = \frac{4\pi^2r^3}{g_pR^2}$$

$$= \frac{4\pi^2 \times 9200000^3}{11{\cdot}2 \times 8600000^2} = 3711482$$

$$T = 6092 \text{ seconds} \qquad \mathbf{1}$$

$$\approx 101{\cdot}5 \text{ minutes}$$

A body of mass 2kg starts from rest at the origin. It moves in a horizontal straight line with acceleration $(6 - 2t)\mathbf{i}$ m s^{-2}, where t is the time in seconds from the start of the motion and **i** is the unit vector in the direction of motion. Find the work done by the total force acting on the body during the interval in which the body moves from rest to its maximum speed. **6**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	6	1.2/4	4	2	1.2.4/2.4.4	1.1.7	1997 SY5 Q4

$$\ddot{\mathbf{r}} = (6 - 2t)\mathbf{i}$$

$$\dot{\mathbf{r}} = (6t - t^2)\mathbf{i} + \mathbf{c}$$

$$\text{when } t = 0,\ \dot{\mathbf{r}} = \mathbf{0} \Rightarrow \mathbf{c} = \mathbf{0}$$

$$\dot{\mathbf{r}} = (6t - t^2)\mathbf{i}$$ **1**

Maximum speed occurs when $\ddot{\mathbf{r}} = \mathbf{0}$

$$\Rightarrow \quad 6 - 2t = 0$$

$$2t = 6 \text{ so } t = 3 \text{ seconds}$$ **1**

$$\text{Work done} = \int_0^3 F\,dx = \int_0^3 F\frac{dx}{dt}\,dt$$

$$= \int_0^3 Fv\,dt$$ **1**

$$\mathbf{F} = m\ddot{\mathbf{r}} = 2(6 - 2t)\mathbf{i}$$ **1**

$$\mathbf{v} = \dot{\mathbf{r}} = (6t - t^2)\mathbf{i}$$

$$W = \int_0^3 2(6 - 2t)(6t - t^2)\,dt$$

$$= \int_0^3 2(36t - 18t^2 + 2t^3)\,dt$$

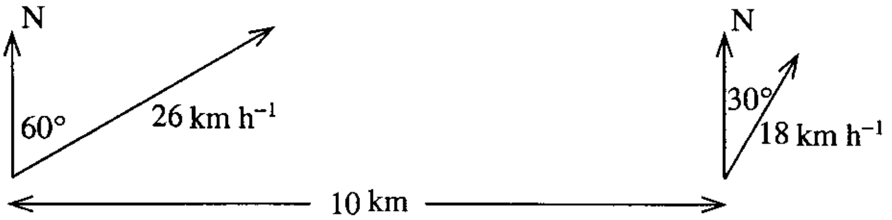
$$= 2\left[18t^2 - 6t^3 + \frac{1}{2}t^4\right]_0^3$$ **1**

$$= 2[162 - 162 + 40{\cdot}5] = 81 \text{ J}$$ **1**

[Using change in KE, perhaps commoner, would then involve 2.4.9 rather than 2.4.4.]

A fishing boat sails from a harbour on a bearing of 060° at a constant speed of 26 kilometres per hour. A cargo ship is sailing on a bearing of 030° at a constant speed of 18 kilometres per hour. The fishing boat leaves the harbour at noon, at which time the cargo ship is 10 kilometres due East of the harbour.
Calculate the time at which the two vessels will be nearest to each other. **7**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	7	1.2	5	2	1.2.7/3	1.2.2	1997 SY5 Q5



$$\mathbf{v}_f = \text{velocity of fishing boat} = 26\sin 60°\mathbf{i} + 26\cos 60°\mathbf{j} = 22{\cdot}52\mathbf{i} + 13\mathbf{j}$$

$$\mathbf{v}_f = \text{velocity of cargo ship} = 18\sin 30°\mathbf{i} + 18\cos 30°\mathbf{j} = 9\mathbf{i} + 15{\cdot}5\mathbf{j}$$ **1**

$$\mathbf{r}_f = (22.52t + c)\mathbf{i} + (13t + d)\mathbf{j}$$

$$\mathbf{r}_c = (9t + e)\mathbf{i} + (15{\cdot}59t + f)\mathbf{j}$$

$$\text{when } t = 0,\ \mathbf{r}_f = \mathbf{0} \quad \text{and} \quad \mathbf{r}_c = 10\mathbf{i}$$

$$\Rightarrow \quad c = d = f = 0 \quad \text{and} \quad e = 10$$

$$\mathbf{r}_f = 22.52t\,\mathbf{i} + 13t\,\mathbf{j} \qquad \mathbf{r}_c = (9t + 10)\mathbf{i} + 15{\cdot}59t\,\mathbf{j}$$ **1,1**

$${}_f\mathbf{r}_c = \text{position of } f \text{ relative to } c$$

$$= \mathbf{r}_f - \mathbf{r}_c = (13{\cdot}52t - 10)\mathbf{i} - 2{\cdot}59t\,\mathbf{j}$$ **1**

$$|{}_f\mathbf{r}_c|^2 = 182{\cdot}79t^2 - 270{\cdot}4t + 100 + 6{\cdot}71t^2$$

$$= 189{\cdot}50t^2 - 270{\cdot}4t + 100$$ **1**

$$\frac{d|{}_f\mathbf{r}_c|^2}{dt} = 379{\cdot}0t - 270{\cdot}4 = 0$$ **1**

$$\Rightarrow t = \frac{270{\cdot}4}{379{\cdot}0} = 0{\cdot}713 \text{ hours} \approx 42{\cdot}8 \text{ minutes}$$

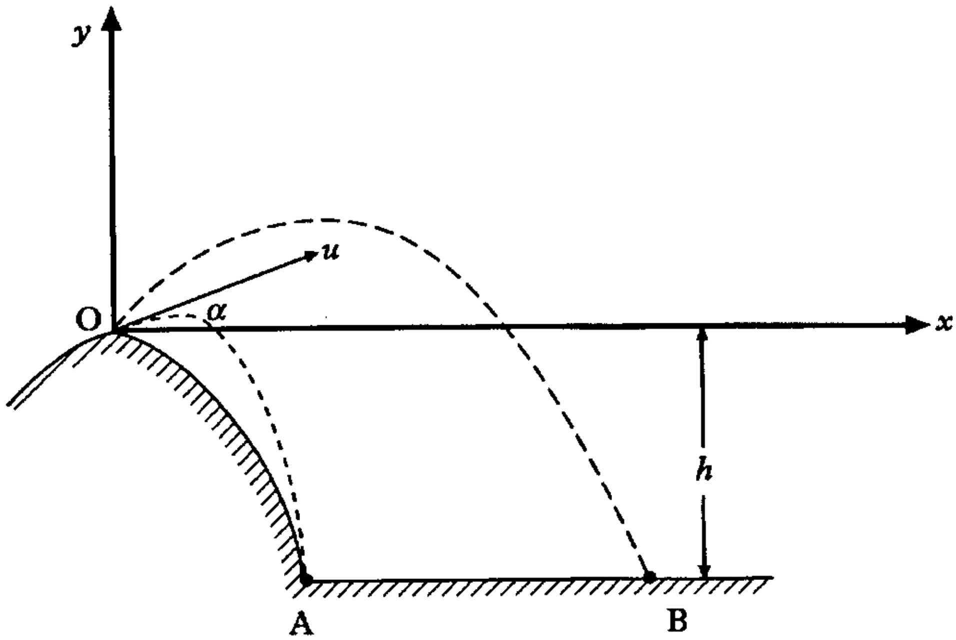
Time of nearest approach is ≈ 12.43 pm. **1**

A particle is projected from a point O at the top of a hill, with speed u at an angle α above the horizontal, and moves freely under constant gravity.

(a) Given that Ox and Oy are rectangular axes as indicated in the diagram below, show that the trajectory of the particle has equation

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha},$$

where g is the magnitude of the acceleration due to gravity. **5**



The hill is modelled by the curve OA, with equation $y = -\frac{x^2}{750}$, and the horizontal plane, AB, below the hill, is modelled by the line $y = -h$, where h is the height of the hill.

(b) A particle is projected from O at an angle α above the horizontal such that $\tan\alpha = \frac{1}{5}$. The initial speed of projection, in metres per second, is $\frac{1}{2}\sqrt{975g}$. The particle hits the plane AB at A. Show that the height of the hill is $83\frac{1}{3}$ metres. **6**

(c) A second particle is projected from O at an angle θ above the horizontal such that $\tan\theta = \frac{2}{3}$. The initial speed of projection, in metres per second, is now $\sqrt{325g}$. The particle hits the plane AB at B. Calculate the distance from A to B. **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	5	1.3	5		1.3.6	1.3.1	1997 SY5 Q6
(b)	6	1.3		6	1.3.9/6		
(c)	4	1.3		4	1.3.9/6		

(a)

$$\ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} = -g\mathbf{j}$$

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$

$$\dot{x} = c \qquad \dot{y} = -gt + d$$

$$\text{when } t = 0, \dot{x} = u\cos\alpha \qquad \text{when } t = 0, \dot{y} = u\sin\alpha$$

$$\Rightarrow c = u\cos\alpha \qquad \Rightarrow d = u\sin\alpha$$

$$\dot{x} = u\cos\alpha \qquad \dot{y} = -gt + u\sin\alpha \qquad \mathbf{1}$$

$$x = ut\cos\alpha + e \qquad y = -\tfrac{1}{2}gt^2 + ut\sin\alpha + f$$

$$\text{when } t = 0, x = 0 \Rightarrow e = 0 \qquad \text{when } t = 0, y = 0 \Rightarrow f = 0$$

$$x = ut\cos\alpha \quad \mathbf{1} \qquad y = ut\sin\alpha - \tfrac{1}{2}gt^2 \qquad \mathbf{1}$$

$$t = \frac{x}{u\cos\alpha} \qquad \mathbf{1}$$

$$y = ut\sin\alpha - \tfrac{1}{2}gt^2$$

$$y = u\frac{x}{u\cos\alpha}\sin\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

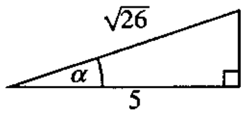
$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \qquad \mathbf{1}$$

(b) Coordinates of A = $(x, y) = (x, -h)$ and $y = \dfrac{-x^2}{750}$

$$\Rightarrow -h = -\frac{x^2}{750} \Rightarrow x = \sqrt{750h}$$

Thus A is $\left(\sqrt{750h}, -h\right)$. **2**

Substituting into



$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \text{ where } \tan\alpha = \frac{1}{5}$$

$$\cos\alpha = \frac{5}{\sqrt{26}} \Rightarrow \cos^2\alpha = \frac{25}{26} \text{ and } u^2 = \frac{975g}{4} \qquad \mathbf{1}$$

Hence

$$-h = \frac{\sqrt{750h}}{5} - \frac{g\,750h}{2 \times \frac{975g}{4} \times \frac{25}{26}} \qquad \mathbf{1}$$

$$= \sqrt{30h} - \frac{78000}{48750}h = \sqrt{30h} - \frac{8}{5}h$$

$$\frac{3}{5}h = \sqrt{30h} \qquad \mathbf{1}$$

$$\frac{9}{25}h^2 = 30h \qquad h \neq 0$$

$$\text{so } h = \frac{30 \times 25}{9} = 83\tfrac{1}{3}\text{ m} \qquad \mathbf{1}$$

(c) Want to find x coordinate when $y = -88\frac{1}{3}$.

This means $u^2 = 325g$ so $\tan\theta = \frac{2}{3} \Rightarrow \cos^2\theta = \frac{9}{13}$.

$$-88\frac{1}{3} = \frac{2}{3}x - \frac{gx^2 \times 13}{2 \times 325g \times 9} \qquad \mathbf{1}$$

$$-88\frac{1}{3} = \frac{2}{3}x - \frac{x^2}{450}$$

$$-37500 = 300x - x^2$$

$$x^2 - 300x - 37500 = 0 \qquad \mathbf{1}$$

$$x = \frac{300}{2} \pm \frac{1}{2}\sqrt{90000 + 4 \times 37500}$$

$$= 150 \pm \frac{1}{2}\sqrt{240000}$$

$$= 150 \pm 245 = -95 \text{ or } 395$$

Must be positive and so is 395. **1**

Previous $x = \sqrt{750 \times 83\frac{1}{3}} = 250$.

So the distance from A to B = $395 - 250 = 145$ metres. **1**

A sledge and rider have a combined mass of 112 kg. In a test run, the sledge and rider are free to move in a straight line down the line of greatest slope of a hill, inclined at angle α to the horizontal, such that $\sin\alpha = 0{\cdot}22$. When moving with speed v m s^{-1} the sledge and rider experience a resistance whose magnitude, measured in newtons, is given by

$$F(v) = 78 + 0{\cdot}28v^2.$$

(a) Find the terminal speed of the sledge and rider. **3**

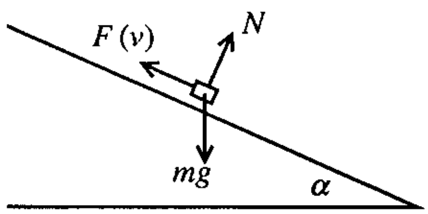
(b) Show that the magnitude of the acceleration of the sledge and rider, in m s^{-2}, is $\frac{1}{400}(V^2 - v^2)$, where V m s^{-1} is the terminal speed. **2**

(c) Show that the sledge and rider will travel a distance $200 \ln \frac{9}{5}$ metres from rest, down the slope, before acquiring a speed of $\frac{2}{3}V$ m s^{-1}. **5**

(d) Show also that the time taken to reach this speed is approximately 13·3 seconds. **5**

$\left[\text{You may assume that } \int \frac{1}{a^2 - v^2}\,dv = \frac{1}{2a}\ln\left|\frac{a+v}{a-v}\right| + c, \text{ where } a > 0.\right]$

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	2.5	3		2.5.5	1.4.1/11	1997 SY5 Q7
(b)	2	2.5	2		2.5.2	1.4.11	
(c)	5	2.5		5	2.5.3	2.5.1	
(d)	5	2.5		5	2.5.3	2.5.1	



(a) Resultant force = $mg \sin\alpha - F(v)$.
Maximum speed is reached when the resultant force is zero.

$$mg \sin\alpha - F(v) = 0$$

$$mg \sin\alpha = 78 + 0{\cdot}28V^2 \qquad \mathbf{1}$$

$$V = \sqrt{\frac{mg \sin\alpha - 78}{0{\cdot}28}}$$

$$= \sqrt{\frac{112 \times 9{\cdot}8 \times 0{\cdot}22 - 78}{0{\cdot}28}} = 24{\cdot}2 \text{ m s}^{-1} \qquad \mathbf{1}$$

The terminal speed is $24{\cdot}2$ m s^{-1}. **1**

(b)

$$ma = F_{RES}$$

$$ma = mg \sin\alpha - F(v)$$

$$ma = mg \sin\alpha - 78 - 0{\cdot}28v^2 \qquad \mathbf{1}$$

$$ma = 0{\cdot}28V^2 - 0{\cdot}28v^2$$

$$a = \frac{0{\cdot}28}{m}(V^2 - v^2) = \frac{0{\cdot}28}{112}(V^2 - v^2) \qquad \mathbf{1}$$

$$a = \frac{1}{400}(V^2 - v^2)$$

(c)

$$a = \tfrac{1}{400}(V^2 - v^2)$$

$$v\frac{dv}{dx} = \frac{1}{400}(V^2 - v^2)$$

$$\int \frac{v\,dv}{(V^2 - v^2)} = \int \frac{1}{400}\,dx \qquad \mathbf{1}$$

$$-\frac{1}{2}\ln|V^2 - v^2| = \frac{x}{400} + c \qquad \mathbf{1}$$

When $x = 0, v = 0 \Rightarrow c = -\tfrac{1}{2}\ln V^2$. **1**

$$-\frac{1}{2}\ln|V^2 - v^2| = \frac{x}{400} - \frac{1}{2}\ln V^2$$

$$\frac{x}{400} = \frac{1}{2}\ln V^2 - \frac{1}{2}\ln|V^2 - v^2|$$

$$x = 200 \ln\left|\frac{V^2}{V^2 - v^2}\right| \qquad \mathbf{1}$$

Given that $v = \tfrac{2}{3}V$ then

$$x = 200 \ln\left|\frac{V^2}{V^2 - \tfrac{4}{9}V^2}\right| = 200 \ln\frac{9}{5} \text{ metres} \qquad \mathbf{1}$$

(d)

$$a = \frac{1}{400}\left(V^2 - v^2\right)$$

$$\frac{dv}{dt} = \frac{1}{400}\left(V^2 - v^2\right)$$

$$\int \frac{1}{V^2 - v^2}dv = \int \frac{1}{400}\,dt \qquad \mathbf{1}$$

$$\frac{1}{2V}\ln\left|\frac{V + v}{V - v}\right| = \frac{t}{400} + c \qquad \mathbf{1}$$

when $t = 0, v = 0 \Rightarrow c = 0$

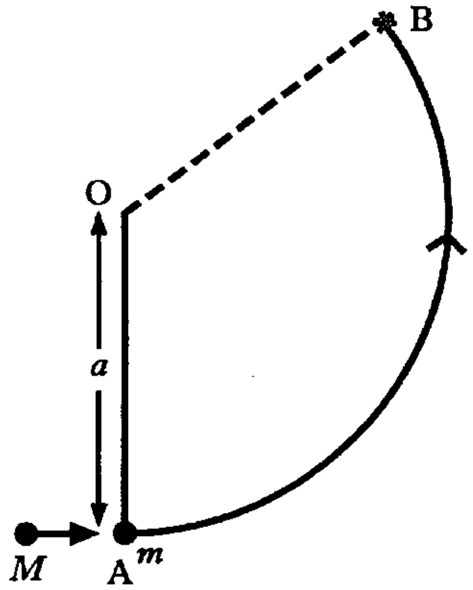
$$t = \frac{200}{V}\ln\left|\frac{V + v}{V - v}\right| \qquad \mathbf{1}$$

Given that $v = \frac{2}{3}V$

$$t = \frac{200}{V}\ln\left|\frac{V + \frac{2}{3}V}{V - \frac{2}{3}V}\right| = \frac{200}{V}\ln\left|\frac{\frac{5}{3}}{\frac{1}{3}}\right| \qquad \mathbf{1}$$

$$= \frac{200}{24{\cdot}2}\ln 5 = 13{\cdot}3 \text{ seconds} \qquad \mathbf{1}$$

A game of "conkers" is modelled by considering a chestnut to be a particle of mass m attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O and the chestnut hangs in equilibrium at A, vertically below O.



The chestnut at A is struck horizontally by a second chestnut of mass M. After the collision, this second chestnut continues horizontally at three fifths of its speed, V, just prior to the collision, while the original chestnut has an initial horizontal speed u.

(a) Use the conservation of linear momentum to find u in terms of V, m and M. **3**

Consider two chestnuts for which $m = 16$ grams and $M = 20$ grams.

(b) Show that the chestnut which was initially in equilibrium will execute a complete circle about O provided that

$$V \geqslant \sqrt{20ga},$$

where g is the magnitude of the acceleration due to gravity. **8**

(c) Given that $V = \sqrt{16ga}$, the tension in the string becomes zero at position B. Show that $\cos AOB = -\frac{2}{3}$ and find the speed of the chestnut at B, in terms of g and a. **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	2.3	3		2.3.3		1997 SY5 Q8
(b)	8	2.4		8	2.4.11	2.4.7/8,2.1.3	
(c)	4	2.4	2	2	2.4.11	2.4.6/2.1.3	

(a)

$$MV + 0 = M\tfrac{3}{5}V + mu \qquad \mathbf{2}$$

$$mu = \frac{2MV}{5}$$

$$u = \frac{2M}{5m}V \qquad \mathbf{1}$$

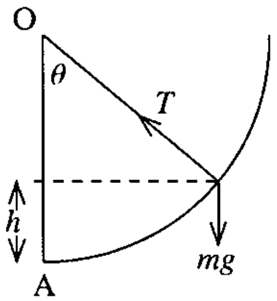
(b) Energy equation

$$E_P + E_U = \text{constant}$$

$$mgh + \tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 + 0 \qquad \mathbf{1}$$

$$mga(1 - \cos\theta) + \tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 \qquad \mathbf{1}$$

$$v^2 = u^2 + 2ga(\cos\theta - 1)$$



Radial equation

$$T - mg\cos\theta = \frac{mv^2}{a} \qquad \mathbf{2}$$

$$T - mg\cos\theta = \frac{m}{a}\left(u^2 + 2ga(\cos\theta - 1)\right)$$

$$= \frac{mu^2}{a} + 2mg(\cos\theta - 1)$$

$$T = \frac{mu^2}{a} + mg(3\cos\theta - 2) \qquad \mathbf{1}$$

Complete circles if $T \geqslant 0$ when $\cos\theta = -1$ ($\theta = 180°$). **1**

$$\frac{mu^2}{a} + mg(3\cos\theta - 2) \geqslant 0$$

$$\frac{mu^2}{a} + mg(-3 - 2) \geqslant 0$$

$$\frac{mu^2}{a} \geqslant 5mg$$

$$u^2 \geqslant 5ga$$

Given that $m = 16$ and $M = 20$

$$u = \frac{2M}{5m}V = \frac{2 \times 20}{5 \times 16}V = \frac{1}{2}V \qquad \mathbf{1}$$

$$\Rightarrow \quad \frac{V^2}{4} \geqslant 5ga$$

$$V^2 \geqslant 20ga$$

$$V^2 \geqslant \sqrt{20ga} \qquad \mathbf{1}$$

(c) If $V^2 = 16ga$ then $u^2 = 4ga$.
Use the radial equation when $T = 0$.

$$\frac{mu^2}{a} + mg(3\cos\theta - 2) = 0 \qquad \mathbf{1}$$

$$4mg + mg(3\cos\theta - 2) = 0$$

$$4 + 3\cos\theta - 2 = 0$$

$$3\cos\theta = -2$$

$$\cos\theta = -\frac{2}{3} \text{ i.e. } \cos \text{AOB} = -\frac{2}{3} \qquad \mathbf{1}$$

Use the energy equation to find the speed at B.

$$v^2 = u^2 + 2ga(\cos\theta - 1)$$

$$= 4ga + 2ga\left(-\frac{2}{3} - 1\right) \qquad \mathbf{1}$$

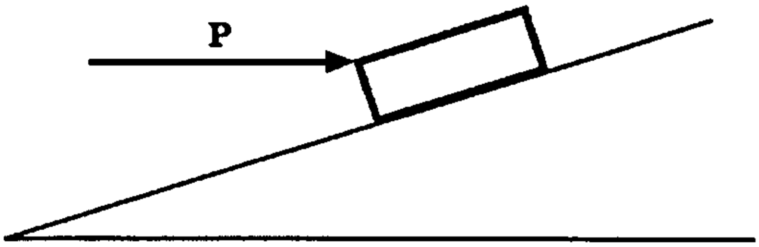
$$= 4ga - \frac{10ga}{3} = \frac{2ga}{3}$$

$$V = \sqrt{\frac{2ga}{3}} \qquad \mathbf{1}$$

A block of wood of mass m is placed on a rough inclined plane, the coefficient of friction between the block and plane being $\frac{3}{10}$.

(a) Find the angle at which the plane must be inclined to the horizontal for the block to be on the point of slipping **down** the plane. **3**

(b) With the plane inclined at this angle, a force, **P**, is applied horizontally to the block as shown in the diagram below.



Given that the block is now on the point of slipping **up** the plane, show that the magnitude of **P** is

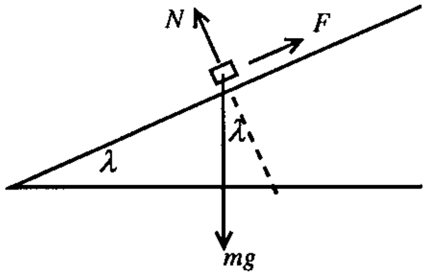
$$\frac{60}{91}mg,$$

where g is the magnitude of the acceleration due to gravity. **6**

(c) With this force **P** still applied horizontally, calculate the size of the angle at which the plane must be inclined to the horizontal for the block to be once more on the point of slipping **down** the plane. **6**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	1.4	3		1.4.3/6	1.4.5	1997 SY5 Q9
(b)	6	1.4		6	1.4.11/3	1.4.6	
(c)	6	1.4	2	4	1.4.11/3	1.4.6	

(a)



Resolving forces parallel to the plane $mg \sin\lambda = F$ (α)

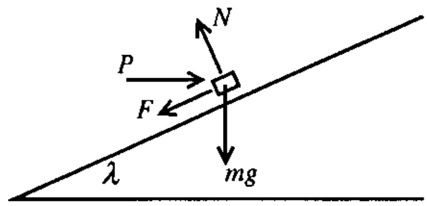
and perpendicular to the plane $mg \cos\lambda = N$ (β) **1**

$$(\alpha) \div (\beta) \Rightarrow \frac{mg \sin\lambda}{mg \cos\lambda} = \frac{F}{N} \Rightarrow \tan\lambda = \frac{F}{N}$$

But in limiting equilibrium $F = \mu N$. **1**

$$\tan\lambda = \frac{\mu N}{N} \Rightarrow \tan\lambda = \mu \Rightarrow \lambda = \tan^{-1}\mu = \tan^{-1}0{\cdot}3 \approx 16{\cdot}7^\circ$$ **1**

(b)



Resolving parallel to plane $P\cos\lambda = mg \sin\lambda + F$ **1**

$\Rightarrow F = P\cos\lambda - mg\sin\lambda$ **1**

Resolving perpendicular to plane $N = mg\cos\lambda + P\sin\lambda$ **1**

$$F = \frac{3}{10}N \Rightarrow \frac{F}{N} = \frac{P\cos\lambda - mg\sin\lambda}{mg\cos\lambda + P\sin\lambda} = \frac{3}{10}$$ **1**

$$\Rightarrow \frac{3}{10} = \frac{P - mg\tan\lambda}{mg + P\tan\lambda} = \frac{P - mg \times 0{\cdot}3}{mg + P \times 0{\cdot}3}$$ **1**

$$\Rightarrow \frac{3}{10} = \frac{10P - 3mg}{10mg + 3P}$$

$$100P - 30mg = 30mg + 9P$$

$$91P = 60mg \Rightarrow P = \frac{60}{91}mg$$ **1**

(c) Resolving parallel to plane $P\cos\alpha = mg\sin\alpha - F$ **1**

Resolving perpendicular to plane $N = mg\cos\alpha + P\sin\alpha$

$F = \frac{3}{10}P$ so dividing

$$\frac{3}{10} = \frac{mg\sin\alpha - P\cos\alpha}{mg\cos\alpha + P\sin\alpha} = \frac{mg\tan\alpha - P}{mg + P\tan\alpha}$$ **1**

$$3mg + 3P\tan\alpha = 10mg\tan\alpha - 10P$$

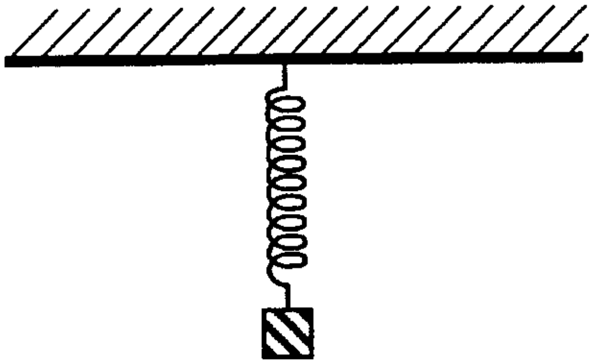
$$P = \frac{60}{91}mg \Rightarrow 3mg + \frac{180}{91}mg\tan\alpha = 10mg\tan\alpha - \frac{600}{91}mg$$ **1**

$$\Rightarrow 273 + 180\tan\alpha = 910\tan\alpha - 600$$

$$273 + 600 = (910 - 180)\tan\alpha$$ **2**

$$\tan\alpha = \frac{873}{730} \Rightarrow \alpha \approx 50{\cdot}1^\circ$$ **1**

Large springs for shock absorbers are tested at a research laboratory. A body of mass 50kg is suspended from a test spring of natural length 0·80 metres, which has the other end attached to a fixed horizontal surface.



(a) Given that, when in equilibrium, the body extends the spring by 0·14 metres, find the modulus of elasticity of the spring. **2**

(b) The body is now pulled 0·20 metres vertically down from its equilibrium position and then released from rest. Take y metres as the vertical displacement of the body from its equilibrium position, t seconds after release. Show that, when all resistive forces are ignored,

$$\frac{d^2y}{dt^2} + 70y = 0.$$ **3**

Hence find an expression for y in terms of t. **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	2	2.2	2		2.2.6	2.2.5	1997 SY5 Q10
(b)	3	2.2		3	2.2.7	1.4.8/1.1.4	
	3	2.2	3		2.2.2/3	2.2.1	

(a)

$$T = \frac{\lambda y}{l} \Rightarrow mg = \frac{\lambda y}{l} \qquad \mathbf{1}$$

$$\lambda = \frac{mgl}{y} = \frac{50 \times 9{\cdot}8 \times 0{\cdot}8}{0{\cdot}14} = 2800 \text{ N} \qquad \mathbf{1}$$

(b)

$$ma = F_{RES}$$

$$m\frac{d^2y}{dt^2} = mg - \frac{\lambda}{l}(0{\cdot}14 + y) \qquad \mathbf{1}$$

$$= -\frac{\lambda}{l}y \qquad \mathbf{1}$$

i.e. $$\frac{d^2y}{dt^2} + \frac{\lambda}{ml}y = 0$$

$$\frac{d^2y}{dt^2} + 70\,y = 0 \qquad \mathbf{1}$$

$$\frac{d^2y}{dx^2} = -70y$$

$\Rightarrow$ SHM with $\omega = \sqrt{70}$ **1**

The mass starts at an extremity so $y = a\cos\omega t$ **1**

$$y = 0{\cdot}20\cos\sqrt{70}\,t \qquad \mathbf{1}$$

A satellite moves with constant speed in a circular orbit around the planet Venus. The satellite is orbiting at a height of 400 kilometres above the surface of Venus, whose radius is 6050 kilometres and whose mass is $4{\cdot}9 \times 10^{24}$ kg.

Given that Newton's inverse square law of gravitation applies and that the Universal Constant of Gravitation, G, is $6{\cdot}67 \times 10^{-11}$ kg^{-1} s^{-2}, calculate the speed of the satellite. **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	2.1	3		2.1.5/6	2.1.7/3	1998 SY5 Q1

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \qquad \mathbf{1}$$

$$v = \sqrt{\frac{GM}{r}}$$

$$= \sqrt{\frac{6{\cdot}67 \times 10^{-11} \times 4{\cdot}9 \times 10^{24}}{6450\,000}} \qquad \mathbf{1}$$

$$= 7118 \text{ m s}^{-1} \qquad \mathbf{1}$$

$$= 7120 \text{ m s}^{-1} \text{ to 3 sig. figures}$$

A train of mass 80 tonnes is travelling at a constant speed of v metres per second along a straight, level track against a resistance whose magnitude is $200v$ newtons. Given that the power at which the train's engine is working is 80kW, find the speed of the train. **2**

What increase in power will be required for the train to maintain the same constant speed up a hill, inclined at $\sin^{-1}\frac{1}{240}$ to the horizontal, against the same resistance? **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	2	2.4	2		2.4.5	1.4.1	1998 SY5 Q2
	3	2.4	3		2.4.5	1.4.11	

(a)

$$F_{\text{RESULTANT}} = \frac{P}{V} - 200V$$

$$0 = \frac{80\,000}{V} - 200V \qquad \mathbf{1}$$

$$200V = \frac{80\,000}{V}$$

$$V^2 = 400 \Rightarrow V = 20 \text{ m s}^{-1} \qquad \mathbf{1}$$

(b)

$$F_{\text{RESULTANT}} = \frac{P}{V} - mg\frac{1}{240} - 200V$$

$$0 = \frac{P}{20} - \frac{80\,000 \times 9{\cdot}8}{240} - 200 \times 20 \qquad \mathbf{1}$$

$$0 = 12P - 784\,000 - 960\,000$$

$$P = \frac{1744\,000}{12} = 145\,333 \text{ W} \qquad \mathbf{1}$$

$$\text{Increase in power} = 65\tfrac{1}{3} \text{ kW} \qquad \mathbf{1}$$

An arrow is fired with initial velocity vector $8\mathbf{i} + 3\mathbf{j} + 25\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors in the directions of orthogonal axes Ox, Oy and Oz respectively. The arrow is then subject to an acceleration $-g\mathbf{k}$, where g is the magnitude of the acceleration due to gravity. At the same instant as the arrow is fired, an apple is projected from a tree with initial velocity vector $3\mathbf{i} - \mathbf{j}$. The apple is also subject to an acceleration $-g\mathbf{k}$. Given that the arrow is fired at time $t = 0$, from the origin, and that the initial position vector of the apple is $15\mathbf{i} + 12\mathbf{j} + 75\mathbf{k}$, show that the arrow strikes the apple and state the time of the collision. **6**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	6	1.2	6		1.2.4/7	1.2.2/6	1998 SY5 Q3

$$\mathbf{a}_a = -g\mathbf{k}$$

$$\mathbf{v}_a = c\mathbf{i} + d\mathbf{j} + (-gt + e)\mathbf{k}$$

$$\text{when } t = 0, \quad \mathbf{v}_a = 8\mathbf{i} + 3\mathbf{j} + 25\mathbf{k}$$

$$\Rightarrow \quad \mathbf{v}_a = 8\mathbf{i} + 3\mathbf{j} + (25 - gt)\mathbf{k}$$ **1**

$$\mathbf{r}_a = (8t + f)\mathbf{i} + (3t + g)\mathbf{j} + \left(25t - \tfrac{1}{2}gt^2 + h\right)\mathbf{k}$$

$$\text{when } t = 0, \quad \mathbf{r}_a = \mathbf{0}$$

$$\Rightarrow \quad \mathbf{r}_a = 8t\mathbf{i} + 3t\mathbf{j} + \left(25t - \tfrac{1}{2}gt^2\right)\mathbf{k}$$ **1**

$$\mathbf{a}_P = -g\mathbf{k}$$

$$\mathbf{v}_P = p\mathbf{i} + q\mathbf{j} + (-gt + r)\mathbf{k}$$

$$\text{when } t = 0 \quad \mathbf{v}_P = 3\mathbf{i} - \mathbf{j}$$

$$\Rightarrow \quad \mathbf{v}_P = 3\mathbf{i} - \mathbf{j} - gt\mathbf{k}$$ **1**

$$\mathbf{r}_P = (3t + s)\mathbf{i} + (-t + u)\mathbf{j} + \left(-\tfrac{1}{2}gt^2 + v\right)\mathbf{k}$$

$$\text{when } t = 0, \quad \mathbf{r}_P = 15\mathbf{i} + 12\mathbf{j} + 75\mathbf{k}$$

$$\Rightarrow \mathbf{r}_P = (3t + 15)\mathbf{i} + (12 - t)\mathbf{j} + \left(75 - \tfrac{1}{2}gt^2\right)\mathbf{k}$$ **1**

Therefore, the velocity of a relative to P, ${}_P\mathbf{r}_a$, is given by

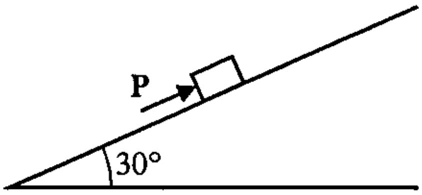
$${}_P\mathbf{r}_a = \mathbf{r}_P - \mathbf{r}_a = \left((3t + 15)\mathbf{i} + (12 - t)\mathbf{j} + \left(75 - \tfrac{1}{2}gt^2\right)\mathbf{k}\right) - \left(8t\mathbf{i} + 3t\mathbf{j} + \left(25t - \tfrac{1}{2}gt^2\right)\mathbf{k}\right)$$

$$= (15 - 5t)\mathbf{i} + (12 - 4t)\mathbf{j} + (75 - 25t)\mathbf{k}$$ **1**

$$= \mathbf{0} \quad \text{when } t = 3$$ **1**

The collision will be after 3 seconds.

A block of wood of mass 5 kg is placed on a rough slope, inclined at an angle of 30° to the horizontal. It is held in position by the action of a force **P** acting parallel to the slope, as shown in the diagram below.

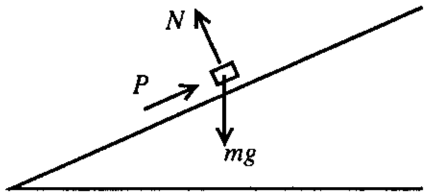


The coefficient of friction between the block of wood and the surface of the slope is $\frac{2}{5}$.

Find the greatest and least values of the magnitude of **P** for the block to remain stationary on the slope. **6**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	6	1.4	6		1.4.3/6	1.4.11/5	1998 SY5 Q4

Friction acts either up or down slope.



Point of slipping down the slope:

Perpendicular to the slope: $N = 5g\cos 30°$ **1**

Parallel to the slope: $P + F = 5g\sin 30°$ **1**

$P + \frac{2}{5}N = 5g\sin 30°$ since limiting equilibrium **1**

Hence $P + \frac{2}{5}5g\cos 30° = 5g\sin 30°$

$P = \frac{5}{2}g - \sqrt{3}\,g$

Least force $P = 7{\cdot}5$ N **1**

Point of slipping up the slope:

Perpendicular to the slope: $N = 5g\cos 30°$

Parallel to the slope: $P = F + 5g\sin 30°$ **1**

$P = \frac{2}{5}5g\cos 30° + 5g\sin 30°$

Hence $P = \frac{5}{2}g + \sqrt{3}\,g$

Greatest force $P = 41{\cdot}5$ N **1**

Thus $7{\cdot}5 \leqslant P \leqslant 41{\cdot}5$.

A snooker ball of mass 150 grams is struck by a snooker cue. The magnitude of the contact force, $F(t)$ newtons, between the cue and the ball can be modelled by the expression,

$$F(t) = 12 \sin(20\pi t), \qquad 0 \leqslant t \leqslant 0.05,$$

where t seconds is the time from the start of the impact.

Given that the ball is initially at rest and assuming no spin is imparted to the ball, calculate the speed of the ball 0·05 seconds after the start of the impact. **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	2.3	2	1	2.3.2		1998 SY5 Q5

$$mv - mu = \int_0^{0\cdot05} 12 \sin 20\pi t \, dt \qquad \mathbf{1}$$

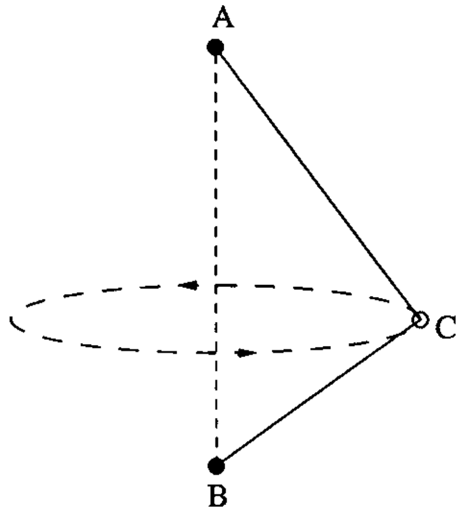
$$mv - 0 = \left[-\frac{12}{20\pi} \cos 20\pi t \right]_0^{0\cdot05} \qquad \mathbf{1}$$

$$mv = -\frac{3}{5\pi}(\cos\pi - \cos 0)$$

$$0\cdot15v = -\frac{3}{5\pi}(-1 - 1)$$

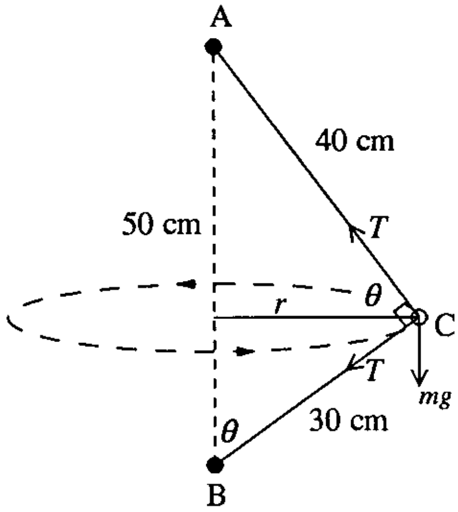
$$v = \frac{6}{5\pi \times 0\cdot15} = 2\cdot55 \text{ m s}^{-1} \qquad \mathbf{1}$$

An inextensible string of length 70 centimetres is attached to two fixed points A and B, where B is 50 centimetres vertically below A. A smooth ring, C, of mass 100 grams, is free to slide on the string. The ring is pulled aside until the string is taut. The ring is then projected so that it moves in a horizontal circle about the line AB, with constant angular speed, as indicated in the diagram.



Given that AC is 40 centimetres, calculate the angular speed of the ring. **7**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	7	2.1	5	2	2.1.3/4	1.4.3	1998 SY5 Q6



$$\cos\theta = \frac{30}{50} = \frac{3}{5}$$
$$\sin\theta = \frac{40}{50} = \frac{4}{5} \qquad \mathbf{1}$$

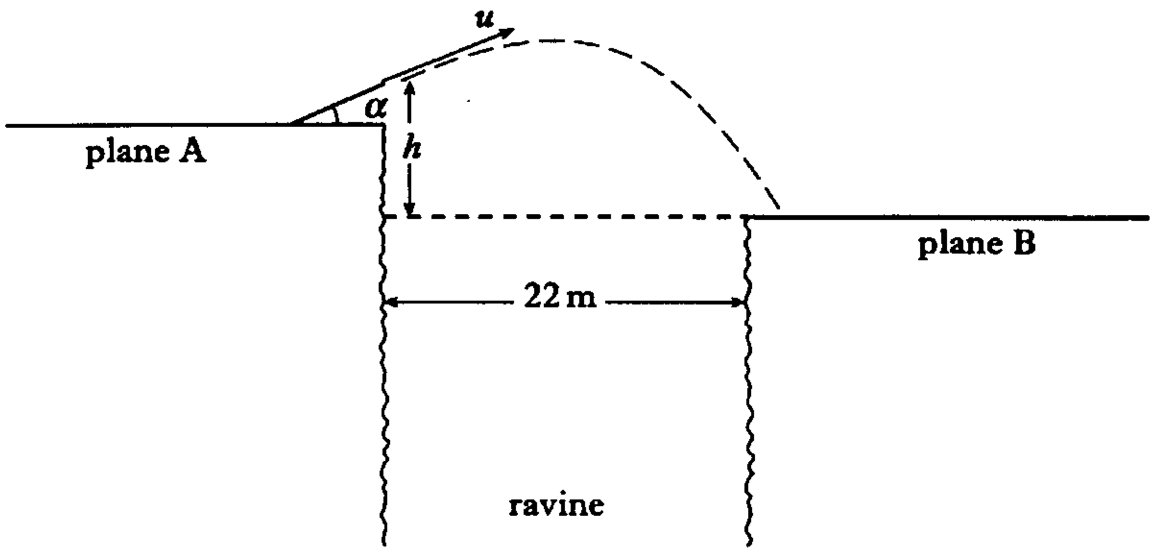
Horizontally

$$T\cos\theta + T\sin\theta = mr\omega^2 \qquad \mathbf{1}$$
$$\frac{3}{5}T + \frac{4}{5}T = mr\omega^2$$
$$T = \frac{5}{7}mr\omega^2 \qquad \mathbf{1}$$

Vertically

$$T\sin\theta = T\cos\theta + mg \qquad \mathbf{1}$$
$$\frac{4}{5}T = \frac{3}{5}T + mg$$
$$\frac{1}{5}T = mg \Rightarrow T = 5mg \qquad \mathbf{1}$$
$$\Rightarrow \quad \frac{5}{7}mr\omega^2 = 5mg$$
$$\omega^2 = \frac{7g}{r}$$
$$\text{but } r = 40\cos\theta = 40 \times \frac{3}{5} = 24 \text{ cm} \qquad \mathbf{1}$$
$$\Rightarrow \quad \omega^2 = \frac{7g}{0{\cdot}24} = 285{\cdot}8$$
$$\omega = 16.9 \text{ rad s}^{-1} \qquad \mathbf{1}$$

A stunt driver is to attempt to clear a 22m wide ravine on a motor cycle. He will drive off a ramp, inclined at angle a to the horizontal, on the edge of plane A, and land on plane B. The driver and cycle are modelled by a particle being projected from the top of the ramp with initial speed u, at angle α to the horizontal, and then falling freely under constant gravity.



(a) Show that the greatest height, H, reached above the top of the ramp is given by

$$H = \frac{u^2 \sin^2 \alpha}{2g},$$

where g is the magnitude of the acceleration due to gravity. **3**

(b) Show that the horizontal range, R, of the driver and cycle is given by

$$R = \frac{u^2 \sin 2\alpha}{2g}\left(1 + \sqrt{1 + \frac{h}{H}}\right),$$

where h is the vertical height from plane B to the top of the ramp on plane A. **6**

(c) Given that $u = 15 \text{ m s}^{-1}$, $\alpha = 30°$ and $h = 4$ m, by how much will the driver and cycle clear the ravine? **3**

(d) At what height above the surface of plane B is the cyclist at the instant when he passes over the edge of plane B? **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	1.3	3		1.3.4	1.3.3	1998 SY5 Q7
(b)	6	1.3		6	1.3.9	1.3.3	
(c)	3	1.3	1	2	1.3.9	1.3.4	
(d)	3	1.3	1	2	1.3.9	1.3.4	

(a) Horizontally: $x = ut\cos\alpha$ Vertically: $y = ut\sin\alpha - \frac{1}{2}gt^2$ **1**

We need to find t when $\dot{y} = 0$.

$$\dot{y} = u\sin\alpha - gt$$
$$0 = u\sin\alpha - gt$$
$$t = \frac{u\sin\alpha}{g} \qquad \mathbf{1}$$

Substituting into the equation for the vertical motion

$$H = u\frac{u\sin\alpha}{g}\sin\alpha - \frac{1}{2}g\frac{u^2\sin^2\alpha}{g^2}$$
$$H = \frac{u^2\sin^2\alpha}{g} - \frac{1}{2}\frac{u^2\sin^2\alpha}{g} = \frac{u^2\sin^2\alpha}{2g} \qquad \mathbf{1}$$

(b) We want to find t when $y = -h$.

$$ut\sin\alpha - \tfrac{1}{2}gt^2 = -h \qquad \mathbf{1}$$
$$\tfrac{1}{2}gt^2 - ut\sin\alpha - h = 0$$
$$t = \frac{u\sin\alpha}{g} \pm \frac{1}{g}\sqrt{u^2\sin^2\alpha + 2gh} \qquad \mathbf{1}$$

The value of t must be positive so

$$t = \frac{u\sin\alpha}{g} + \frac{1}{g}\sqrt{u^2\sin^2\alpha + 2gh} \qquad \mathbf{1}$$
$$= \frac{1}{g}\left(u\sin\alpha + u\sin\alpha\sqrt{1 + \frac{2gh}{u^2\sin^2\alpha}}\right)$$
$$= \frac{u\sin\alpha}{g}\left(1 + \sqrt{1 + \frac{h}{H}}\right) \qquad \mathbf{1}$$
$$R = ut\cos\alpha = \frac{u^2\sin\alpha\cos\alpha}{g}\left(1 + \sqrt{1 + \frac{h}{H}}\right) \qquad \mathbf{1}$$
$$R = \frac{u^2\sin 2\alpha}{2g}\left(1 + \sqrt{1 + \frac{h}{H}}\right) \qquad \mathbf{1}$$

(c)

$$H = \frac{u^2\sin^2\alpha}{2g} = \frac{225 \times \sin^2 30^\circ}{2 \times 9{\cdot}8} \approx 2{\cdot}87 \text{ m} \qquad \mathbf{1}$$
$$R = \frac{u^2\sin 2\alpha}{2g}\left(1 + \sqrt{1 + \frac{h}{H}}\right)$$
$$= \frac{225\sin 60^\circ}{2 \times 9{\cdot}8}\left(1 + \sqrt{1 + \frac{4}{2{\cdot}87}}\right) \approx 25{\cdot}32 \text{ m} \qquad \mathbf{1}$$

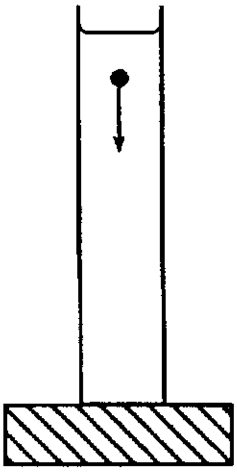
The motorcyclist will clear the ravine by about 3·32 metres. **1**

(d)

$$x = ut\cos\alpha$$
$$t = \frac{x}{u\cos\alpha} = \frac{22}{15\cos 30^\circ} \approx 1{\cdot}69 \text{ seconds} \qquad \mathbf{1}$$
$$y = ut\sin\alpha - \frac{1}{2}gt^2$$
$$= 15 \times 1{\cdot}69 \times \sin 30^\circ - 4{\cdot}9 \times 1{\cdot}69^2 = -1{\cdot}32 \text{ m} \qquad \mathbf{1}$$

Height above plane B = 4 − 1.32 = 2.68 metres. **1**

A body of mass m kilograms falls through a column of clear viscous oil.



The forces acting on the body are its weight, an upward thrust equal to the weight of oil that the body displaces and a resistive force of magnitude $8mv$ newtons, where v metres per second is the vertical speed of the body as it is falling through the oil.

(a) Given that the weight of oil displaced is two-thirds of the weight of the body, show that an equation of motion for the body is

$$v\frac{dv}{dy} = \frac{g}{3} - 8v,$$

where y metres is the vertical distance travelled through the oil and g is the magnitude of the acceleration due to gravity in m s^{-2}. **2**

Hence find the terminal speed of the body. **2**

(b) Given that the body is released from rest below the surface of the oil, find the vertical distance travelled through the oil by the body before it reaches 95% of its terminal speed. **7**

(c) Find also the time taken, from rest, for the body to reach 95% of its terminal speed. **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	2	2.5		2	2.5.2	2.5.1	1998 SY5 Q8
	2	2.5	2		2.5.5		
(b)	7	2.5	1	6	2.5.3		
(c)	4	2.5	1	3	2.5.3	2.5.1	

(a)

$$F = m_b g - m_0 g - 8m_b v \qquad 1$$

$$m_b v \frac{dv}{dy} = m_b g - \frac{2}{3} m_b g - 8m_b v$$

$$v \frac{dv}{dy} = \frac{g}{3} - 8v \qquad 1$$

For terminal speed, the acceleration is zero.

$$0 = \tfrac{g}{3} - 8v \qquad 1$$

$$v = \frac{g}{24} \approx 0{\cdot}41 \text{ m s}^{-1} \qquad 1$$

(b)

$$3v \frac{dv}{dy} = g - 24v$$

$$\int \frac{3v}{g - 24v}\, dv = \int dy \qquad 1$$

$$\int \left[-\frac{1}{8} + \frac{g}{8}\left(\frac{1}{g - 24v} \right) \right] dv = \int dy \qquad 1$$

$$-\frac{1}{8}v - \frac{g}{192} \ln|g - 24v| = y + c \qquad 1$$

$$\text{when } y = 0,\ v = 0 \Rightarrow c = -\frac{g}{192} \ln|g| \qquad 1$$

$$y = \frac{g}{192} \ln\left| \frac{g}{g - 24v} \right| - \frac{1}{8}v \qquad 1$$

$$\text{95\% of terminal speed} = \frac{19}{20} \times \frac{g}{24}$$

$$= \frac{19g}{480} \approx 0{\cdot}3979 \text{ m s}^{-1} \qquad 1$$

$$\text{Hence } y = \frac{g}{192} \ln\left| \frac{g}{g - \frac{19g}{20}} \right| - \frac{19g}{3840}$$

$$= \frac{g}{192} \ln 20 - \frac{19g}{3840}$$

$$= \frac{g}{192}\left(\ln 20 - \frac{19}{20} \right) \approx 0{\cdot}10 \text{ m} \qquad 1$$

(c)

$$3 \frac{dv}{dt} = g - 24v$$

$$\int \frac{3}{g - 24v} dv = \int dt \qquad 1$$

$$-\frac{1}{8} \ln|g - 24v| = t + c \qquad 1$$

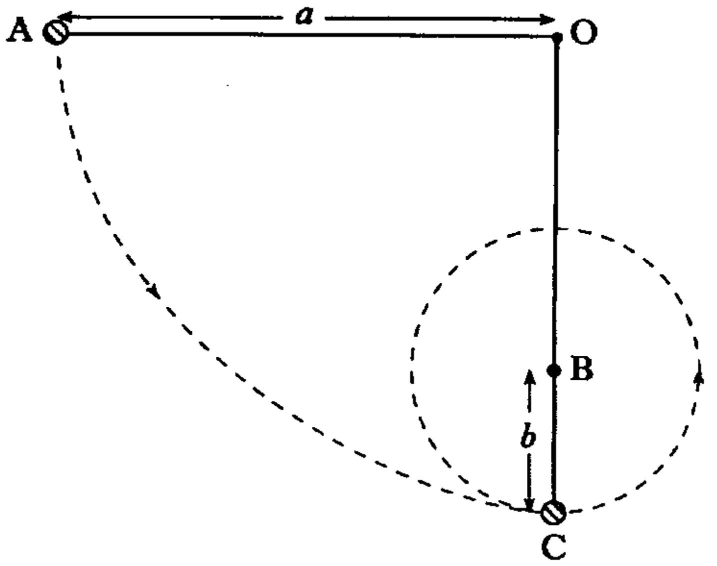
$$\text{when } t = 0,\ v = 0 \Rightarrow c = -\frac{1}{8} \ln g$$

$$t = \frac{1}{8} \ln\left| \frac{g}{g - 24v} \right| \qquad 1$$

$$\text{when } v = \frac{19g}{480} \qquad t = \frac{1}{8} \ln\left| \frac{g}{g - \frac{19g}{20}} \right|$$

$$= \tfrac{1}{8} \ln 20 \approx 0{\cdot}37 \text{ seconds} \qquad 1$$

A particle of mass m is attached to one end of a light, inextensible string of length a. The other end of the string is fixed at position O. The particle is held at A, with the string taut and horizontal, and released from rest to move in a vertical plane. At a point B, vertically below O, a thin, smooth rod is fixed perpendicular to the plane of motion. When the particle is at the point C, vertically below O, the string strikes the rod. This causes the particle to move in a smaller circle of radius b, about the thin rod.



(a) In the subsequent motion show that the tension, T, in the string is given by

$$T = mg(3\cos\theta - 2) + \frac{2mga}{b},$$

where θ is the angle between the string and BC and g is the magnitude of the acceleration due to gravity. **5**

(b) Show that the particle will complete a vertical circle about B if

$$b \leqslant \frac{2}{5}a,$$ **2**

(c) Now consider the particular case where $b = \frac{1}{2}a$.

(i) Show that, at the instant when the tension in the string becomes zero,

$$\cos\theta = -\frac{2}{3},$$

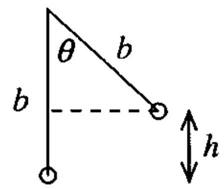
where θ is as in part (a). **3**

(ii) Find, in terms of a, the greatest height above C reached by the particle after the string becomes slack. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	5	2.4		5	2.4.11	2.4.6/3	1998 SY5 Q9
(b)	2	2.4	2		2.4.11		
(c)(i)	3	2.4	3		2.4.11		
(c)(ii)	5	2.4		5	2.4.11	1.3.4	

(a) Energy equation:

$$\frac{1}{2}mv^2 + mgh = \text{a constant}$$



From the diagram

$$h = b - b\cos\theta = b(1 - \cos\theta) \qquad \mathbf{1}$$

$$\frac{1}{2}mv^2 + mgb(1 - \cos\theta) = mga \qquad \mathbf{1}$$

$$v^2 = 2ga - 2gb(1 - \cos\theta) \qquad \mathbf{1}$$

Radial equation

$$T - mg\cos\theta = \frac{mv^2}{b} \qquad \mathbf{1}$$

$$v^2 = \frac{b}{m}T - bg\cos\theta$$

Equating these expressions for v^2

$$\frac{b}{m}T - bg\cos\theta = 2ga - 2gb + 2gb\cos\theta$$

$$\frac{b}{m}T = gb(3\cos\theta - 2) + 2ga$$

$$T = mg(3\cos\theta - 2) + \frac{2mga}{b} \qquad \mathbf{1}$$

(b) For a complete circle, $T \geqslant 0$ for $0 \leqslant \theta \leqslant 180°$

$$\Rightarrow \quad mg(3\cos 180° - 2) + \frac{2mga}{b} \geqslant 0 \qquad \mathbf{1}$$

$$\frac{2mga}{b} \geqslant 5mg$$

$$2a \geqslant 5b$$

$$b \leqslant \frac{2}{5}a \qquad \mathbf{1}$$

(c) (i) Using the expression for T obtained in part (a) when $T = 0$ and $b = \frac{1}{2}a$ **1**
gives

$$mg(3\cos\theta - 2) + \frac{2mga}{b} = 0 \qquad \mathbf{1}$$

$$3\cos\theta - 2 + 4 = 0$$

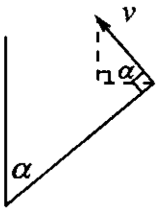
$$\cos\theta = -\frac{2}{3} \qquad \mathbf{1}$$

(ii) Applying the expression for v^2 from part (a) when $b = \frac{1}{2}a$ and $\cos\theta = -\frac{2}{3}$

$$v^2 = 2ga - 2gb(1 - \cos\theta)$$

$$v^2 = 2ga - ga\left(1 - \left(-\frac{2}{3}\right)\right)$$

$$v^2 = 2ga - \frac{5}{3}ga = \frac{1}{3}ga \qquad \mathbf{1}$$



Vertical component of velocity = $v \sin\alpha$
where $\cos\alpha = \frac{2}{3}$ (and so $\sin\alpha = \frac{\sqrt{5}}{3}$). **1**
Let h be the vertical height gained after $T = 0$.
Energy conservation gives

$$mgh = \tfrac{1}{2}mv^2 \sin^2\alpha$$

$$h = \frac{v^2 \sin^2\alpha}{2g}$$

$$= \frac{\frac{ga}{3} \times \frac{5}{9}}{2g} = \frac{5}{54}a \qquad \mathbf{1}$$

Height above B $= b\cos\alpha + h$

$$= \frac{a}{2} \times \frac{2}{3} + \frac{5}{54}a$$

$$= \frac{1}{3}a + \frac{5}{54}a = \frac{23}{54}a \qquad \mathbf{1}$$

Height above C $= \frac{1}{2}a + \frac{23}{54}a = \frac{25}{27}a$ **1**

A mass m is suspended between two light springs AP and PB, of natural lengths $2a$ and a respectively, as shown in the diagram below.

The two ends A and B are fixed, with B a distance $4a$ vertically below A. The mass rests in equilibrium at position P. Spring AP has modulus of elasticity mg and spring PB has modulus of elasticity $3mg$, where g is the magnitude of the acceleration due to gravity.

(a) Given that x_1 is the extension in spring AP and x_2 is the compression in spring PB, show that

$$x_1 - x_2 = a.$$ **2**

(b) Show also that the ratio of the lengths of AP and PB is given by

$$\frac{AP}{PB} = \frac{11}{3}.$$ **7**

(c) Show that the potential energy, E_p, stored in a spring is given by

$$E_p = \frac{1}{2}\frac{\lambda}{l}x^2,$$

where λ is the modulus of elasticity of the spring, l is its natural length and x is the extension or compression of the spring. **3**

(d) Find an expression, in terms of m, g and a, for the total potential energy stored in the two springs. **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	2	2.2	2		2.2.8		1998 SY5 Q10
(b)	7	2.2		7	2.2.6/8		
(c)	3	2.4		3	2.4.8		
(d)	3	2.4	2	1	2.4.8		

(a) $$(2a + x_1) + (a - x_2) = 4a$$ **1**

$$3a + x_1 - x_2 = 4a$$

$$x_1 - x_2 = a$$ **1**

(b) $$T_1 + T_2 = mg$$ **1**

$$\frac{\lambda_1 x_1}{l_1} + \frac{\lambda_2 x_2}{l_2} = mg$$

$$\frac{mg\,x_1}{2a} + \frac{3mg\,x_2}{a} = mg$$ **2**

Hence $x_1 + 6x_2 = 2a$ **1**

and from above $x_1 - x_2 = a$

so $7x_2 = a$

$$x_2 = \frac{a}{7} \text{ and } x_1 = a + x_2 = \frac{8a}{7}$$ **1**

$$\frac{\text{AP}}{\text{PB}} = \frac{2a + x_1}{a - x_2}$$ **1**

$$= \frac{2a + \frac{8}{7}a}{a - \frac{1}{7}a}$$

$$= \frac{\frac{22}{7}a}{\frac{6}{7}a} = \frac{11}{3}$$ **1**

(c) $$E_P = \int_0^X \frac{\lambda}{l} x\,dx$$ **1**

$$= \left[\frac{1}{2}\frac{\lambda}{l}x^2\right]_0^X$$ **1**

$$= \frac{1}{2}\frac{\lambda}{l}X^2 - 0 = \frac{1}{2}\frac{\lambda}{l}X^2$$ **1**

(d) $$E_P = \frac{1}{2}\frac{\lambda_1}{l_1}x_1^2 + \frac{1}{2}\frac{\lambda_2}{l_2}x_2^2$$

$$= \frac{1}{2}\frac{mg}{2a}x_1^2 + \frac{1}{2}\frac{3mg}{a}x_2^2$$ **1**

$$= \frac{mg}{4a}\frac{64a^2}{49} + \frac{3mg}{2a}\frac{a^2}{49}$$ **1**

$$= \frac{mga}{4}\left(\frac{64 + 6}{49}\right) = \frac{mga}{4} \times \frac{70}{49} = \frac{5}{14}mga$$ **1**

A particle of unit mass executes simple harmonic motion about a fixed point O on a straight line, such that its displacement x from O at time t is given by the equation

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

where ω is a positive constant.

(a) Show that its speed v at displacement x from O is given by the equation

$$v^2 = \omega^2(a^2 - x^2),$$

where a is the amplitude of the motion. **3**

(b) Given that its speed is $\sqrt{5}$ m s^{-1} when its displacement from the centre of oscillation is $\frac{2}{3}a$ metres, and the maximum magnitude of the acceleration is 18 m s^{-2}, calculate the period and amplitude of the motion. **6**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	2.5		3	2.5.4	2.5.1	1998 SY5 Q11
(b)	6	2.2	4	2	2.5.4	2.5.1,2.2.8	

(a)

$$\ddot{x} = -\omega^2 x$$

$$v\frac{dv}{dx} = -\omega^2 x \qquad \mathbf{1}$$

$$\int v\,dv = \int -\omega^2 x\,dx$$

$$\tfrac{1}{2}v^2 = -\tfrac{1}{2}\omega^2x^2 + c \qquad \mathbf{1}$$

$$\text{when } v = 0,\ a = 0 \Rightarrow c = \tfrac{1}{2}\omega^2a^2$$

$$\tfrac{1}{2}v^2 = -\tfrac{1}{2}\omega^2x^2 + \tfrac{1}{2}\omega^2a^2$$

$$v^2 = \omega^2(a^2 - x^2) \qquad \mathbf{1}$$

(b)

$$v^2 = \omega^2(a^2 - x^2)$$

$$5 = \omega^2\left(a^2 - \frac{4a^2}{9}\right) \qquad \mathbf{1}$$

$$5 = \frac{5}{9}a^2\omega^2$$

$$a^2\omega^2 = 9 \Rightarrow a\omega = 3 \qquad \mathbf{1}$$

$$a\omega^2 = 18 \qquad \mathbf{1}$$

$$\Rightarrow \quad \frac{a\omega^2}{a\omega} = \frac{18}{3} \Rightarrow \omega = 6 \qquad \mathbf{1}$$

$$\text{and} \quad a = \frac{3}{\omega} = \frac{3}{6} = \frac{1}{2}$$

i.e. the amplitude = 0·5 metres **1**

and the period = $\frac{2\pi}{\omega} = \frac{\pi}{3}$ seconds. **1**

A particle moves along the x-axis with velocity, measured in m s^{-1} given by

$$\mathbf{v} = 2t(3 - t)\mathbf{i},$$

where $\mathbf{i}$ is the unit vector in the positive direction of the x-axis and t is the time in seconds from the start of the motion. Calculate the distance travelled from the start of the motion until the particle is again at rest and find the acceleration at the instant of coming to rest. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	1.1,1.2	5		1.1.7	1.1.4,1.2.4	1999 SY5 Q1

Particle at rest when $t = 3$. **1**

$$\frac{dx}{dt} = 6t - 2t^2$$

$$x = 3t^2 - \frac{2}{3}t^3 + c$$

when $t = 0, x = 0 \Rightarrow c = 0.$

So $x = 3t^2 - \frac{2}{3}t^3.$ **1**

When $t = 3, \quad x = 3 \times 9 - \frac{2}{3} \times 27$

$$= 27 - 18$$

$$= 9 \text{ metres}.$$ **1**

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$= (6 - 4t)\mathbf{i}$$ **1**

When $t = 3, \quad a = 6 - 12$

$$= -6 \text{ m s}^{-2}$$ **1**

A particle describes simple harmonic motion. At an instant one eighth of a period after the particle has been in an extreme position its speed is 2 m s^{-1} and its acceleration has magnitude 8 m s^{-2}. Find the period and the amplitude of the motion. **6**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	6	2.2	1	5	2.2.3/4	2.2.1	1999 SY5 Q2

$$x = a\omega \cos \omega t$$

$$\dot{x} = -a\omega \sin \omega t$$

$$\ddot{x} = -a\omega^2 \cos \omega t \qquad \mathbf{1}$$

When $\omega t = \frac{1}{8} \times 2\pi = \frac{\pi}{4}$ **1**

$$2 = a\omega \sin \tfrac{\pi}{4} \quad \text{and} \quad 8 = a\omega^2 \cos \tfrac{\pi}{4} \qquad \mathbf{1}$$

$$a\omega = 2\sqrt{2} \qquad a\omega^2 = 8\sqrt{2}$$

$$\frac{a\omega^2}{a\omega} = \frac{8\sqrt{2}}{2\sqrt{2}} \qquad \mathbf{1}$$

$$\omega = 4$$

$$\text{Period,} \quad T = \frac{2\pi}{\omega} = \frac{\pi}{2} \text{ seconds} \qquad \mathbf{1}$$

$$\text{Amplitude,} \quad a = \frac{2\sqrt{2}}{\omega} = \frac{1}{\sqrt{2}} \approx 0{\cdot}707 \text{ m} \qquad \mathbf{1}$$

Two aeroplanes P and Q, travelling with constant velocities, are observed on radar monitoring equipment from a communications centre. The communications centre is taken as the origin, O, of a system of orthogonal axes Ox with unit vector **i**, Oy with unit vector **j** and Oz with unit vector **k**. At the instant of observation, the position vectors of P and Q are noted as $9200\mathbf{i} + 3860\mathbf{j} + 2870\mathbf{k}$ and $12800\mathbf{i} + 2660\mathbf{j} + 3350\mathbf{k}$ respectively. Doppler radar records the velocity of P to be $68\mathbf{i} + 21\mathbf{j} - 7\mathbf{k}$ and the velocity of Q to be $53\mathbf{i} + 26\mathbf{j} - 9\mathbf{k}$, referred to the same axes. Distances are measured in metres and velocities in metres per second. Show that the aeroplanes are on a collision course. **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	4	1.2	4		1.2.4/7	1.2.1	1999 SY5 Q3

$\mathbf{v}_P = 68\mathbf{i} + 21\mathbf{j} - 7\mathbf{k}$ $\qquad$ $\mathbf{v}_Q = 53\mathbf{i} + 26\mathbf{j} - 9\mathbf{k}$

$\mathbf{r}_P = (68t + 9200)\,\mathbf{i} + (21t + 3860)\,\mathbf{j} + (-7t + 2870)\,\mathbf{k}$ **1**

$\mathbf{r}_Q = (53t + 12800)\,\mathbf{i} + (26t + 2660)\,\mathbf{j} + (-9t + 3350)\,\mathbf{k}$ **1**

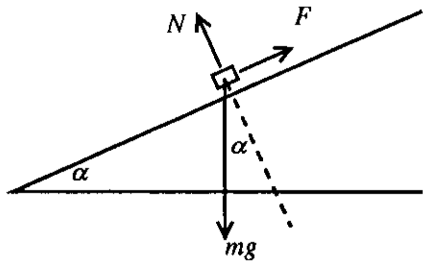
$${}_P\mathbf{r}_Q = \mathbf{r}_P - \mathbf{r}_Q$$

$= (15t - 3600)\,\mathbf{i} + (-5t + 1200)\,\mathbf{j} + (2t - 480)\,\mathbf{k}$ **1**

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$ when $t = 240$ seconds. **1**

The coefficient of friction between a body of mass 4 kg and a plane surface is 0·2. The plane surface is inclined at an angle to the horizontal so that the body is on the point of slipping **down** the plane. With the plane still inclined at this angle, a force is now applied to the body, acting up the line of greatest slope of the plane. Given that the body is then on the point of moving **up** the plane, calculate the magnitude of this applied force. **6**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	6	1.4	6		1.4.3/4	1.4.11	1999 SY5 Q4



$F = mg \sin\alpha$

$\mu N = mg \sin\alpha$ **1**

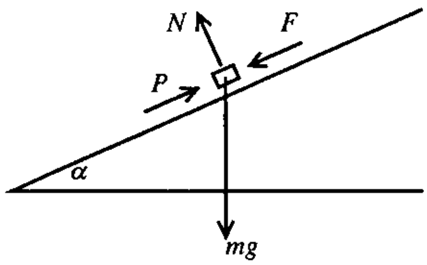
But $\mu = \tan\alpha$

so $\tan\alpha\, N = mg \sin\alpha$

thus $N = mg \cos\alpha$ **1**

$\alpha = \tan^{-1}\mu$

$= \tan^{-1}0{\cdot}2 \approx 11{\cdot}3°$ **1**



$P = F + mg \sin\alpha$ **1**

$P = \mu N + mg \sin\alpha$

$P = \mu mg \cos\alpha + mg \sin\alpha$ **1**

$P = 0{\cdot}2 \times 4 \times 9{\cdot}8 \times \cos 11{\cdot}3° + 4 \times 9{\cdot}8 \times \sin 11{\cdot}3°$

$= 15{\cdot}4$ newtons **1**

5. A snooker cue ball is moving with velocity (5**i** + **j**) m s^{-1}, referred to suitable rectangular axes. It collides with a stationary red ball of equal mass which moves off with velocity (3**i** + 2·5**j**) m s^{-1} referred to the same axes.

(a) Given that no spin is imparted to either ball, use the conservation of linear momentum to calculate the resulting velocity of the cue ball. **3**

(b) Calculate the angle between the velocities of the red ball and the cue ball after impact. **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	2.3	3		2.3.3	2.3.4	1999 SY5 Q5
(b)	2	2.3		2	2.3.4	1.2.5	

(a)

$$m(5\mathbf{i} + \mathbf{j}) + m(0\mathbf{i} + 0\mathbf{j}) = m(v_{x_1}\mathbf{i} + v_{y_1}\mathbf{j}) + m(3\mathbf{i} + 2{\cdot}5\mathbf{j}) \quad \mathbf{1}$$

$$(5\mathbf{i} + \mathbf{j}) = (v_{x_1}\mathbf{i} + v_{y_1}\mathbf{j}) + (3\mathbf{i} + 2{\cdot}5\mathbf{j})$$

$$v_{x_1} = 5 - 3 = 2 \quad \mathbf{1}$$

$$v_{y_1} = 1 - 2{\cdot}5 = -1{\cdot}5 \quad \mathbf{1}$$

$$\mathbf{v} = (2\mathbf{i} - 1{\cdot}5\mathbf{j})\ \text{m s}^{-1}$$

(b)

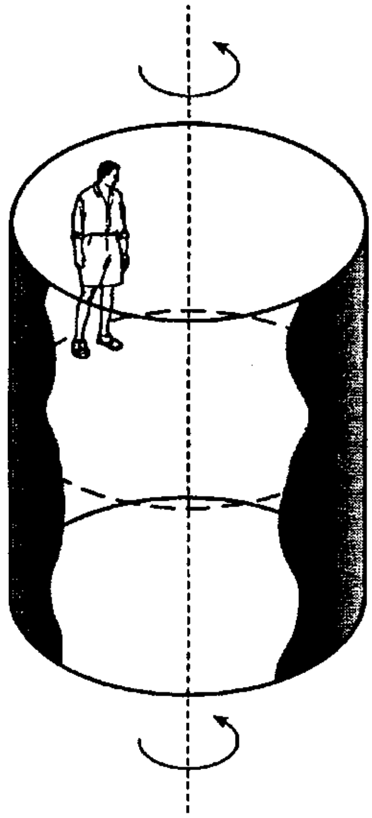
$$\cos\theta = \frac{\mathbf{v}_1.\mathbf{v}_2}{|\mathbf{v}_1||\mathbf{v}_2|}$$

$$= \frac{(2\mathbf{i} - 1{\cdot}5\mathbf{j})(3\mathbf{i} - 2{\cdot}5\mathbf{j})}{\sqrt{6.25}\sqrt{15.25}} \quad \mathbf{1}$$

$$= 0{\cdot}2304663$$

Thus $\theta \approx 76.7°.$ **1**

An amusement park ride consists of an upright drum of diameter 5 metres, inside which people stand with their backs against the wall, as in the diagram below. The drum is spun at increasing speed about its central vertical axis until it reaches a certain steady speed. The floor is then pulled downwards and the occupants do not fall but remain "pinned" against the wall of the drum.



Given that the coefficient of friction between an occupant's clothing and the drum wall is 0·35, what is the minimum angular speed of the drum at which the floor can be pulled downwards safely? **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	4	2.1	4		2.1.3/4	1.4.6	1999 SY5 Q6

$$F = mg = \mu N \qquad 1$$

$$N = mr\omega^2 \qquad 1$$

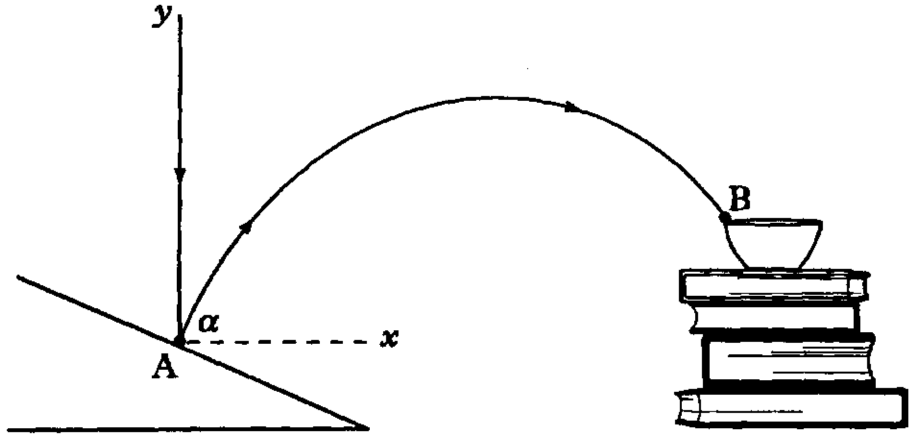
Hence $\mu mr\omega^2 = mg$

$$\omega = \sqrt{\frac{g}{\mu r}} \qquad 1$$

$$= \sqrt{\frac{9{\cdot}8}{2{\cdot}5 \times 0{\cdot}35}}$$

$$\approx 3{\cdot}35 \text{ rad s}^{-1} \qquad 1$$

A "bouncy ball" is dropped vertically onto a hard plane surface as in the diagram below. The ball strikes the plane surface at the point A and is then projected, at angle α to the horizontal, towards a bowl which is placed on a pile of books.



The rim of the bowl is 0·40 metres vertically higher than A and the nearest part of the rim, B, is 1·60 metres horizontally from A. The ball leaves the plane surface with an initial speed of u metres per second. The ball is modelled throughout as a particle.

(a) Taking A as the origin, referred to rectangular axes Ax and Ay, show that an equation describing the trajectory of the particle after striking the plane is given by

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha},$$

where g is the magnitude of the acceleration due to gravity. **4**

(b) Given that $u = 5{\cdot}6\ \mathrm{m\,s^{-1}}$, show that there are two values of α for which the particle just reaches B.

[You may assume that $\dfrac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$.] **4**

(c) For the same value of u and given that the particle reaches B, calculate:

(i) the maximum height the particle can reach above the level of A, **5**

(ii) the minimum time of flight from A to B. **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	1.3	4		1.3.6	1.3.1	1999 SY5 Q7
(b)	4			4	1.3.6/9		
(c)(i)	5			5	1.3.4		
(c)(ii)	2		1	1	1.3.4		

(a)

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$

$$\dot{x} = u\cos\alpha \qquad \dot{y} = -gt + u\sin\alpha$$

$$x = ut\cos\alpha \qquad y = ut\sin\alpha - \tfrac{1}{2}gt^2$$ **2**

$$t = \frac{x}{u\cos\alpha}$$ **1**

$$y = u\sin\alpha\frac{x}{u\cos\alpha} - \tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$ **1**

(b) $y = 0{\cdot}4,\ x = 1{\cdot}6$ and $u = 5{\cdot}6$, so

$$0{\cdot}4 = 1{\cdot}6\tan\alpha - \frac{9{\cdot}8 \times 1{\cdot}6^2}{2 \times 5{\cdot}6^2\cos^2\alpha}$$ **1**

$$0{\cdot}4 = 1{\cdot}6\tan\alpha - 0{\cdot}4(1 + \tan^2\alpha)$$

$$\tan^2\alpha - 4\tan\alpha + 2 = 0$$ **1**

$$\tan\alpha = 2 \pm \sqrt{2}$$

$$\alpha \approx 73{\cdot}7° \text{ or } 30{\cdot}4°$$ **2**

(c) (i) $\dot{y} = u\sin\alpha - gt$

For maximum height $\dot{y} = 0$. **1**

$$\Rightarrow 0 = u\sin\alpha - gt$$

$$t = \frac{u\sin\alpha}{g}$$ **1**

Applying $y = u\sin\alpha t - \frac{1}{2}gt^2$ vertically gives

$$H = \frac{u^2\sin^2\alpha}{g} - \tfrac{1}{2}g\frac{u^2\sin^2\alpha}{g^2} = \frac{u^2\sin^2\alpha}{2g}$$ **1**

$$H = \frac{5{\cdot}6^2\sin^2 73{\cdot}7°}{2 \times 9{\cdot}8} \approx 1{\cdot}47 \text{ m}$$ **1**

$$\left[\text{or } H = \frac{5{\cdot}6^2\sin^2 30{\cdot}4°}{2 \times 9{\cdot}8} \approx 0{\cdot}41 \text{ m}\right]$$

Greatest height is 1·47 metres. **1**

(ii) $$t = \frac{x}{u\cos\alpha} = \frac{1{\cdot}6}{5{\cdot}6\cos 30{\cdot}4°}$$ **1**

So $t \approx 0{\cdot}33$ seconds. **1**

A train of mass 200 tonnes is travelling along a straight horizontal track. The magnitude of the resistance to the motion of the train is $2000 + 200v$ newtons, where v is the speed of the train in metres per second.

(a) Given that the power at which the train's engine is working is 400 kW, find the maximum speed of the train. **4**

(b) Find the time taken for the train to accelerate from 10 m s^{-1} to 20 m s^{-1}. **6**

[You may assume that $\dfrac{v}{(50+v)(40-v)} = \dfrac{1}{9}\left(\dfrac{4}{40-v} - \dfrac{5}{50+v}\right)$.]

(c) In what time will the train come to rest if the power is shut off when the train is travelling at 20 m s^{-1}? **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	2.4	2	2	2.4.5	1.4.1	1999 SY5 Q8
(b)	6	2.5	1	5	2.5.2/3	2.5.5	
(c)	5	2.5	1	4	2.5.2/3		

(a) $F = \frac{P}{v} = \frac{400\,000}{v}.$ **1**

$$F_{RES} = \frac{400000}{v} - 2000 - 200v$$

Maximum speed when $F_{RES} = 0$.

$$\frac{400000}{v} - 2000 - 200v = 0$$ **1**

$$2000 - 10v - v^2 = 0$$

$$(40 - v)(50 + v) = 0$$ **1**

$$v = 40 \text{ m s}^{-1}$$ **1**

(b)

$$200000\frac{dv}{dt} = \frac{400000}{v} - 2000 - 200v$$ **1**

$$1000\frac{dv}{dt} = \frac{2000 - 10v - v^2}{v}$$

$$\int \frac{1000v}{(50 + v)(40 - v)}dv = \int dt$$ **1**

$$\frac{1000}{9}\int\left(\frac{4}{40 - v} - \frac{5}{50 + v}\right)dv = \int dt$$

$$\frac{1000}{9}(-4\ln|40 - v| - 5\ln|50 + v|) = t + c$$ **1**

$$\text{when } t = 0, v = 10 \Rightarrow c = \frac{1000}{9}(-4\ln 30 - 5\ln 60)$$ **1**

$$t = \frac{1000}{9}\left(4\ln\left|\frac{30}{40 - v}\right| + 5\ln\left|\frac{60}{50 + v}\right|\right)$$

$$\text{when } v = 20 \qquad t = \frac{1000}{9}\left(4\ln\frac{3}{2} + 5\ln\frac{6}{7}\right)$$ **1**

$$\approx 94{\cdot}6 \text{ seconds}$$ **1**

(c)

$$200\,000\frac{dv}{dt} = -2000 - 200v$$

$$1000\frac{dv}{dt} = -10 - v$$ **1**

$$\int -\frac{1000}{v + 10}dv = \int dt$$ **1**

$$-1000\ln|v + 10| = t + c$$ **1**

$$\text{when } t = 0, v = 20 \Rightarrow c = -1000\ln 30$$ **1**

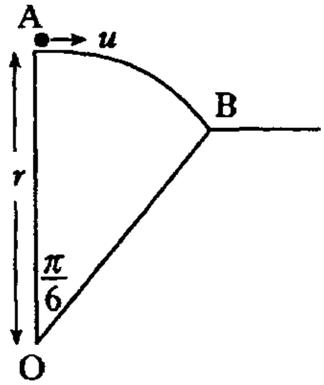
$$t = 1000\ln\frac{30}{v + 10}$$

$$\text{when } v = 0, \qquad t = 1000\ln 3$$

$$\approx 1099 \text{ seconds} = 18 \text{ minutes } 19 \text{ seconds}$$ **1**

Consider a car crossing a hump-backed bridge.

Let the car be modelled by a particle, of mass m, which has speed u at the highest point on the bridge. Assume that the effects of friction and air resistance are negligible. Let the downhill side of the bridge be modelled by the arc of a sector of a circle of radius r, subtending an angle of $\frac{\pi}{6}$ radians at the centre, as in the diagram below.



(a) Show that the greatest speed the car can have at the top of the bridge, without leaving the road surface on the downhill side of the bridge, is given by

$$u = \sqrt{\frac{gr}{2}(3\sqrt{3} - 4)},$$

where g is the magnitude of the acceleration due to gravity. **6**

(b) (i) Given that $u = \sqrt{\frac{3gr}{4}}$ and that the car leaves the surface of the bridge at the point C on arc AB, calculate the angle AOC. **3**

(ii) Given that r is 25 metres, calculate the speed of the car at C. **3**

(c) Given that r is again 25 metres and that the bridge is symmetrical about the vertical through its highest point, calculate the maximum speed that the car could have at the start of the uphill side of the bridge without leaving the surface of the bridge on the downhill side. **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	6	2.4		6	2.4.11	2.4.6/2.1.3	1999 SY5 Q9
(b)(i)	3	2.1	2	1	2.1.3		
(b)(ii)	3	2.4	2	1	2.4.11		
(c)	3	2.4		3	2.4.11	2.4.7/8	

(a)

$$mg\cos\theta - N = \frac{mv^2}{r} \qquad \mathbf{1}$$

$$\tfrac{1}{2}mv^2 + mgh = \tfrac{1}{2}mu^2 \qquad \mathbf{1}$$

$$\tfrac{1}{2}mv^2 + mgr(\cos\theta - 1) = \tfrac{1}{2}mu^2 \qquad \mathbf{1}$$

Eliminating the velocity gives

$$N = mg(3\cos\theta - 2) - \frac{mu^2}{r} \qquad \mathbf{1}$$

To maintain contact with the surface $N \geqslant 0$ when $\theta = \frac{\pi}{6}$

$$mg\left(3\cos\tfrac{\pi}{6} - 2\right) - \tfrac{mu^2}{r} \geqslant 0 \qquad \mathbf{1}$$

$$u^2 \leqslant gr\left(3\cos\tfrac{\pi}{6} - 2\right).$$

Maximum value when $u_{\max}^2 = gr\left(3\cos\frac{\pi}{6} - 2\right)$

$$\text{i.e.}\ u_{\max} = \sqrt{\frac{gr}{2}(3\sqrt{3} - 4)} \qquad \mathbf{1}$$

(b) (i)

$$0 = mg\left(3\cos\theta - 2\right) - \frac{mu^2}{r}$$

$$0 = 3mg\cos\theta - 2mg - \tfrac{3}{4}mg \qquad \mathbf{1}$$

$$\cos\theta = \frac{11}{12} \Rightarrow \theta = 23{\cdot}6° \qquad \mathbf{1,1}$$

(ii)

$$\tfrac{1}{2}mv^2 + mgr(\cos\theta - 1) = \tfrac{1}{2}mu^2$$

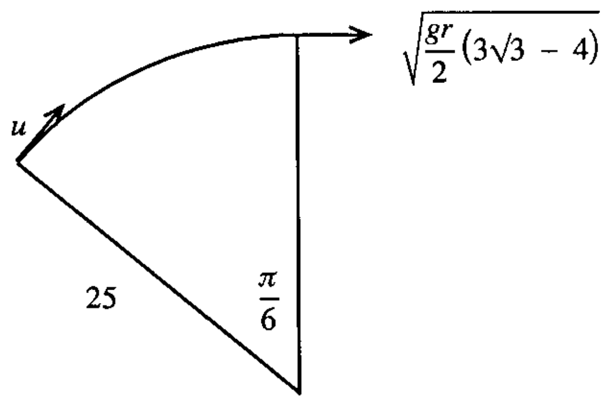
$$v^2 = u^2 - 2gr(\cos\theta - 1)$$

$$= \frac{3gr}{4} - 2gr\left(\frac{11}{12} - 1\right) \qquad \mathbf{1}$$

$$= \frac{3gr}{4} + \frac{2gr}{12} = \frac{11gr}{12} \qquad \mathbf{1}$$

$$\Rightarrow v \approx 15{\cdot}0\ \text{m s}^{-1} \qquad \mathbf{1}$$

(c)



$$\tfrac{1}{2}mu^2 + mg\,25\cos\tfrac{\pi}{6} = \tfrac{1}{2}m\,\tfrac{25g}{2}(3\sqrt{3} - 4) + mg\,25 \qquad \mathbf{2}$$

$$u^2 = 50g\left(1 - \cos\tfrac{\pi}{6}\right) + \tfrac{25g}{2}(3\sqrt{3} - 4)$$

$$\approx 65{\cdot}648 + 146{\cdot}529$$

$$u \approx \sqrt{212{\cdot}2} \approx 14{\cdot}6\ \text{m s}^{-1} \qquad \mathbf{1}$$

A body of constant mass is projected into the gravitational field surrounding the Earth which has radius R metres. Assume that the only force acting on the body is that due to the inverse square law of gravitation.

(a) Consider the case where the body is projected vertically upwards from the surface of the Earth.

(i) Show that an equation of motion for the body is

$$v\frac{dv}{dx} = \frac{-gR^2}{x^2},$$

where v metres per second is the speed of the body, x metres is the vertical height of the body above the **centre** of the Earth and g metres per second per second is the magnitude of the acceleration due to gravity near the surface of the Earth. **3**

(ii) Given that the maximum vertical height of the body above the **surface** of the Earth is h metres, show that the body's initial speed, in metres per second, is given by

$$\sqrt{2gR\left(1 - \frac{R}{R + h}\right)}$$

and calculate the magnitude of the escape velocity for the body, given that $R = 6{\cdot}4 \times 10^6$ metres. **7**

(b) Now consider the case where the body is in a stable circular orbit around the Earth at a height of $3R$ metres vertically above the **surface** of the Earth. Calculate the speed of the body and the period of orbit of the body at this height. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)(i)	3	2.1	3		2.1.5	2.5.1	1999 SY5 Q10
(a)(ii)	7	2.5		7	2.5.3	2.5.5	
(b)	5	2.1	2	3	2.1.6/5	2.1.3/7	

(a) (i) From the law of gravitation $F = -\frac{km}{x^2}$. **1**

When on the surface of the Earth

$$-mg = -\frac{km}{R^2}$$

$$k = gR^2 \qquad \mathbf{1}$$

$$F = -\frac{mgR^2}{x^2}$$

$$mv\frac{dv}{dx} = -\frac{mgR^2}{x^2}$$

$$v\frac{dv}{dx} = -\frac{gR^2}{x^2} \qquad \mathbf{1}$$

(ii)

$$\int v\,dv = \int -\frac{gR^2}{x^2}\,dx \qquad \mathbf{1}$$

$$\tfrac{1}{2}v^2 = \frac{gR^2}{x} + c \qquad \mathbf{1}$$

when $v = u, \quad x = R \quad \Rightarrow c = \tfrac{1}{2}u^2 - gR$ **1**

$$\tfrac{1}{2}v^2 = \frac{gR^2}{x} + \tfrac{1}{2}u^2 - gR$$

when $v = 0, \quad x = R + h$

$$0 = \frac{gR^2}{R+h} + \tfrac{1}{2}u^2 - gR \qquad \mathbf{1}$$

$$\tfrac{1}{2}u^2 = gR\left(1 - \frac{R}{R+h}\right) \qquad \mathbf{1}$$

$$u = \sqrt{gR\left(1 - \frac{R}{R+h}\right)} \qquad \mathbf{1}$$

When $h = \infty$ and $R = 6\cdot4 \times 10^6$ metres

$$|\text{escape velocity}| = \sqrt{2 \times 9\cdot8 \times 6\cdot4 \times 10^5} \approx 11200 \text{ m s}^{-1}. \qquad \mathbf{1}$$

(b)

$$\frac{mv^2}{r} = \frac{mgR^2}{r^2} \qquad \mathbf{1}$$

$$v^2 = \frac{gR^2}{r}$$

$$v = \sqrt{\frac{gR^2}{4R}} = \sqrt{\frac{gR}{4}} \qquad \mathbf{1}$$

$$= \sqrt{\frac{9\cdot8 \times 6\cdot4 \times 10^6}{4}} \approx 3960 \text{ m s}^{-1} \qquad \mathbf{1}$$

$$T = \frac{2\pi R}{v} = \frac{2\pi\, 4R}{3960}$$

$$= \frac{2\pi \times 4 \times 6\cdot4 \times 10^6}{3960} \qquad \mathbf{1}$$

$= 40619$ seconds

$\approx$ 11 hrs 17 min (or 11·3 hrs) **1**

A curling stone is set in motion across horizontal ice with an initial speed of 5 m s^{-1}. Given that the coefficient of friction between the curling stone and the ice is 0·06, calculate how far the stone travels before coming to rest. **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	4	1.4,1.1	4		1.4.6	1.4.5	2000 SY5 Q1
					1.4.8	1.4.9	
					1.1.5		

Dynamic friction so $F = \mu N$

$$F = \mu mg \qquad \mathbf{1}$$

$$a = -\frac{F}{m} = -\frac{\mu mg}{m}$$

$$= -\mu g \qquad \mathbf{1}$$

$$= -0{\cdot}06 \times 9{\cdot}8\,[= -0{\cdot}588 \text{ m s}^{-2}]$$

$$v^2 = u^2 + 2as$$

$$0 = 5^2 + 2 \times (-0{\cdot}588)s \qquad \mathbf{1}$$

$$s = \frac{25}{1{\cdot}176} = 21{\cdot}3 \text{ m} \qquad \mathbf{1}$$

A piston moves with simple harmonic motion. It performs 50 complete oscillations per second and has a maximum speed of 40 m s^{-1}. Calculate the amplitude of the motion. **3**

Find the speed of the piston when the displacement from the centre of oscillation is 0·08 m. **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	2.2	3		2.2.4	2.2.1	2000 SY5 Q2
	2	2.2	2		2.2.4		

$$f = \frac{3000}{60} = 50 \text{ Hz}$$

$$\omega = 2\pi f = 100\pi \qquad \mathbf{1}$$

$$a\omega = 40 \qquad \mathbf{1}$$

$$a = \frac{40}{100\pi} = 0{\cdot}127 \text{ m} \qquad \mathbf{1}$$

$$v^2 = \omega^2(a^2 - x^2)$$

$$= 40^2 - (100\pi)^2 \times 0{\cdot}08^2 \qquad \mathbf{1}$$

$$= 1600 - 64\pi^2 = 968$$

$$v = 31{\cdot}1 \text{ m s}^{-1} \qquad \mathbf{1}$$

A satellite orbiting the Earth is required to have a period of two hours. Assume that the orbit is circular and that the only force acting on the satellite is that due to Newton's inverse square law of gravitation. Given that the radius of the Earth is 6380 kilometres, calculate the required altitude of the satellite above the **surface** of the Earth. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	2.1	5		2.1.6/5	2.1.3/7	2000 SY5 Q3

$$\frac{GMm}{r^2} = mr\omega^2 \qquad \mathbf{1}$$

$$r^3 = \frac{GM}{\omega^2} = \frac{gR^2}{\omega^2} \qquad \mathbf{1}$$

$$r = \sqrt[3]{\frac{gR^2T^2}{4\pi^2}}$$

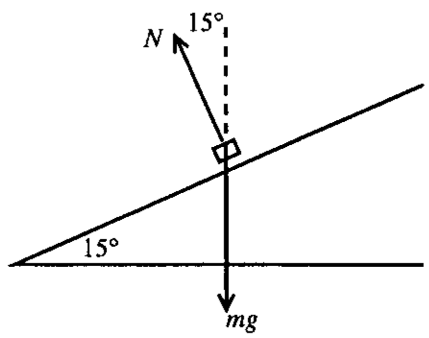
$$= \sqrt[3]{\frac{9{\cdot}8 \times 6380000^2 \times (2 \times 60 \times 60)^2}{4\pi^2}} \qquad \mathbf{1}$$

$$= 8\,060\,000 \text{ m} = 8060 \text{ km} \qquad \mathbf{1}$$

$$h = r - R \approx 8060 - 6380 = 1680 \text{ km} \qquad \mathbf{1}$$

A cyclist travels around a circular section of track which has a radius of curvature of 60 m and is banked at 15° to the horizontal. Given that the cyclist keeps the cycle at right angles to the track's surface, calculate the cyclist's speed at which there is no tendency to slip sideways. **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	4	2.1	4		2.1.4	2.1.3	2000 SY5 Q4
						1.4.3	



Resolving vertically $\quad N\cos 15° = mg$ **1**

Resolving horizontally $\quad N\sin 15° = \dfrac{mv^2}{r}$ **1**

$$\frac{N\sin 15°}{N\cos 15°} = \frac{mv^2}{mgr}$$

$$\tan 15° = \frac{v^2}{gr}$$

$v^2 = gr\tan 15°$ **1**

$= 60 \times 9{\cdot}8 \times \tan 15°$

$= 157{\cdot}6$

$v = 12{\cdot}6 \text{ m s}^{-1}$ **1**

The point of application of the force $\mathbf{F} = 4\mathbf{i} + 11\mathbf{j}$ moves in a straight line from the point $8\mathbf{i} + 5\mathbf{j}$ to the point $32\mathbf{i} + 14\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are mutually perpendicular unit vectors. Given that the force is measured in newtons and the displacement in metres, calculate the work done. **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	2,4	3		2.4.3		2000 SY5 Q5

$$\mathbf{s} = \begin{pmatrix}32\\14\end{pmatrix} - \begin{pmatrix}8\\5\end{pmatrix} = \begin{pmatrix}24\\9\end{pmatrix} \Rightarrow \mathbf{s} = 24\mathbf{i} + 9\mathbf{j} \qquad \mathbf{1}$$

$$W = \mathbf{F.s}$$

$$= (4\mathbf{i} + 11\mathbf{j}).(24\mathbf{i} + 9\mathbf{j}) \qquad \mathbf{1}$$

$$= 96 + 99 = 195 \text{ J} \qquad \mathbf{1}$$

A particle P is projected so that its position vector is given by $6t\mathbf{i} - 3t\mathbf{j} + (55t - 5t^2)\mathbf{k}$, where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are the unit vectors in the directions of the orthogonal axes Ox, Oy and Oz respectively, and t is the time in seconds from the start of the motion. Another particle Q is projected from the point $100\mathbf{i} - 20\mathbf{j} + 47\mathbf{k}$ with initial velocity $-5\mathbf{i} + 50\mathbf{k}$, with respect to the same axes. Distances are measured in metres and velocities in metres per second.

Given that particle Q is subject to the same acceleration as particle P, calculate:

(a) the velocity of P relative to Q; **4**

(b) the position vector of P relative to Q; **2**

(c) the time taken from the start of the motion until the particles are closest together. **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	1.2	4		1.2.4/2	1.2.5/1	2000 SY5 Q6
(b)	2	1.2	2		1.2.4/2	1.2.5/1	
(c)	3	1.2		3	1.2.7	1.2.5	

(a) $\mathbf{r}_P = 6t\mathbf{i} - 3t\mathbf{j} + (55t - 5t^2)\mathbf{k}$

$\mathbf{v}_P = 6\mathbf{i} - 3\mathbf{j} + (55 - 10t)\mathbf{k}$ **1**

$\mathbf{a}_P = -10\mathbf{k}$ **1**

$\mathbf{a}_Q = -10\mathbf{k}$

$\mathbf{v}_Q = c\mathbf{i} + d\mathbf{j} + (-10t + e)\mathbf{k}$

when $t = 0, \mathbf{v}_Q = -5\mathbf{i} + 50\mathbf{k} \Rightarrow c = -5, d = 0$ and $e = 50$

$\mathbf{v}_Q = -5\mathbf{i} + (50 - 10t)\mathbf{k}$ **1**

$${}_P\mathbf{v}_Q = \mathbf{v}_P - \mathbf{v}_Q = (6\mathbf{i} - 3\mathbf{j} + (55 - 10t)\mathbf{k}) - (-5\mathbf{i} + (50 - 10t)\mathbf{k})$$

$$= 11\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$ **1**

(b) $\mathbf{r}_Q = (-5t + f)\mathbf{i} + g\mathbf{j} + (-5t^2 + 50t + h)\mathbf{k}$

when $t = 0, \mathbf{r}_Q = 100\mathbf{i} - 20\mathbf{j} + 47\mathbf{k} \Rightarrow f = 100, g = -20$ and $h = 47$

$\mathbf{r}_Q = (100 - 5t)\mathbf{i} - 20\mathbf{j} + (47 + 50t - 5t^2)\mathbf{k}$ **1**

$${}_P\mathbf{r}_Q = \mathbf{r}_P - \mathbf{r}_Q = (6t\mathbf{i} - 3t\mathbf{j} + (55t - 5t^2)\mathbf{k}) - ((100 - 5t)\mathbf{i} - 20\mathbf{j} + (47 + 50t - 5t^2)\mathbf{k})$$

$$= (11t - 100)\mathbf{i} + (-3t + 20)\mathbf{j} + (5t - 47)\mathbf{k}$$ **1**

(c) $${}_P\mathbf{v}_Q . {}_P\mathbf{r}_Q = 0$$ **1**

$$(11\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}).((11t - 100)\mathbf{i} + (-3t + 20)\mathbf{j} + (5t - 47)\mathbf{k}) = 0$$ **1**

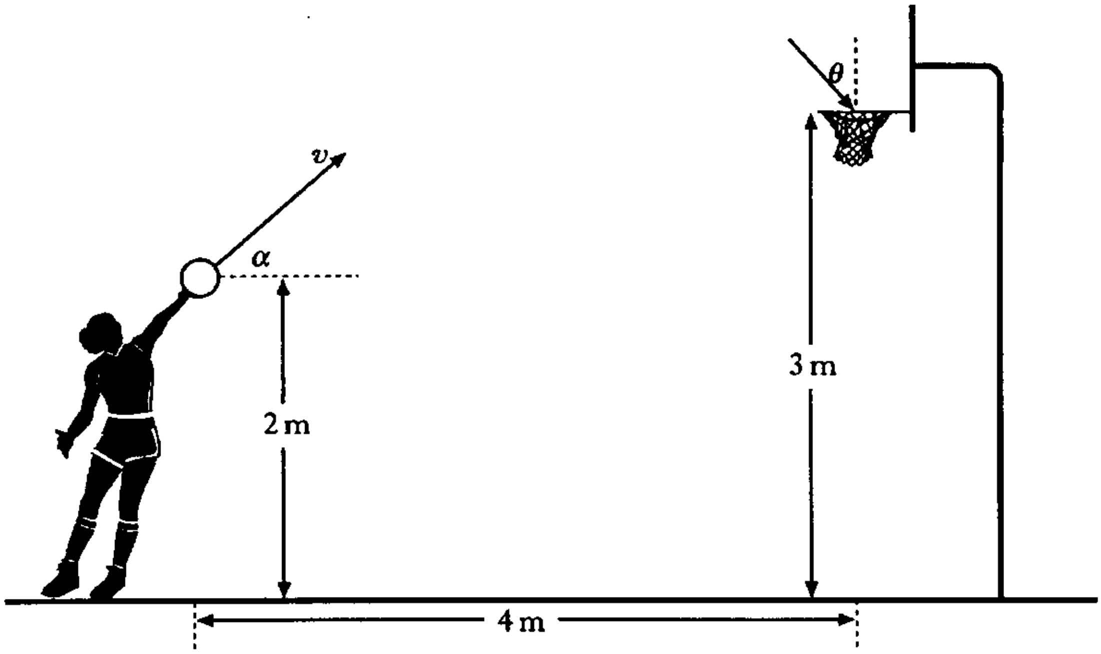
$$121t - 1100 + 9t - 60 + 25t - 235 = 0$$

$$155t = 1395$$

$$t = 9\text{ s}$$ **1**

Alternatives, e.g. using $\frac{d}{dt}|\mathbf{r}_P - \mathbf{r}_Q|^2$, are perfectly acceptable.

A ball is to be thrown from a height of 2 m vertically above the floor into the centre of a net which is 3 m vertically above the floor and 4 m horizontally from where the ball is thrown, as indicated in the diagram below.



Assume that the ball can be modelled by a particle moving under constant gravity and that frictional forces are negligible.

(a) The ball is projected, at angle α to the horizontal, with initial speed v metres per second. Given that $\tan \alpha > \frac{1}{4}$, show that

$$v = \sqrt{\frac{8g}{(4 \tan \alpha - 1) \cos^2 \alpha}},$$

where g m s^{-2} is the magnitude of the acceleration due to gravity. **6**

(b) Given that $\alpha = 45°$, calculate:

(i) the maximum vertical height reached above the floor; **4**

(ii) the value of θ, the angle to the vertical at which the ball drops into the centre of the net. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	6	1.3		6	1.3.3/6	1.3.1	2000 SY5 Q7
(b)(i)	4	1.3	4		1.3.4		
(ii)	5	1.3		5	1.3.9	1.3.4	

(a)

$$\dot{y} = -gt + v\sin\alpha \qquad 1$$

$$x = vt\cos\alpha \qquad y = -\tfrac{1}{2}gt^2 + vt\sin\alpha + 2 \qquad 1$$

$$t = \frac{x}{v\cos\alpha} \qquad y = -\frac{1}{2}g\frac{x^2}{v^2\cos^2\alpha} + v\sin\alpha\frac{x}{v\cos\alpha} + 2 \qquad 1$$

$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha} + 2 \qquad 1$$

$$3 = 4\tan\alpha - \frac{g \times 4^2}{2v^2\cos^2\alpha} + 2 \qquad 1$$

$$\frac{8g}{v^2\cos^2\alpha} = 4\tan\alpha - 1$$

$$v = \sqrt{\frac{8g}{\cos^2\alpha\,(4\tan\alpha - 1)}} \qquad 1$$

(b) (i)

$$v = \sqrt{\frac{8g}{\frac{1}{2}(4-1)}} = \sqrt{\frac{16g}{3}} = 7{\cdot}2\ \text{m s}^{-1} \qquad 1$$

For maximum height $\dot{y} = 0$

$$-gt + v\sin\alpha = 0$$

$$t = \frac{v\sin\alpha}{g} \qquad 1$$

$$y = vt\sin\alpha - \tfrac{1}{2}gt^2 + 2$$

$$H = v\frac{v\sin\alpha}{g}\sin\alpha - \frac{1}{2}g\frac{v^2\sin^2\alpha}{g^2} + 2$$

$$= \frac{v^2\sin^2\alpha}{2g} + 2 \qquad 1$$

$$= \frac{\frac{16g}{3} \times \frac{1}{2}}{2g} + 2 = \frac{4}{3} + 2 = 3\tfrac{1}{3}\ \text{m} \qquad 1$$

(ii)

$$t = \frac{x}{v\cos\alpha}$$

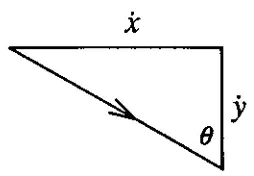
Time of flight, $T = \dfrac{4}{v\cos\alpha} = \dfrac{4}{\sqrt{\frac{16g}{3}}\,\frac{1}{\sqrt{2}}} = \sqrt{\dfrac{6}{g}} = 0{\cdot}8$ s 1

Components of velocity at net

$$\dot{x} = v\cos\alpha = \sqrt{\frac{16g}{3}}\,\frac{1}{\sqrt{2}} = \sqrt{\frac{8g}{3}} = 5{\cdot}112 \qquad 1$$

$$\dot{y} = v\sin\alpha - gt = \sqrt{\frac{8g}{3}} - g\sqrt{\frac{6}{g}} = \sqrt{\frac{8g}{3}} - \sqrt{6g}$$

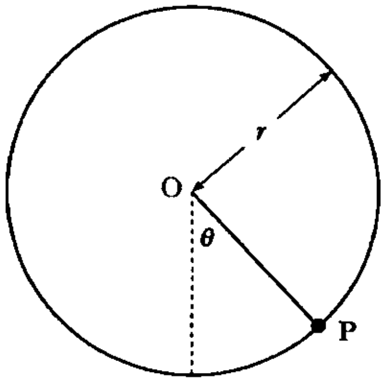
$$\approx 5{\cdot}112 - 7.668 = -2.556 \qquad 1$$



$$\tan\theta = \frac{\dot{x}}{\dot{y}} = \frac{5{\cdot}112}{2{\cdot}556} = 2 \qquad 1$$

$$\theta = 63{\cdot}4° \qquad 1$$

A particle P is attached to one end of a light inextensible string. The other end of the string is fixed at position O. The particle moves in a vertical circle of radius r about O. The angle θ is defined to be the angle between OP and the **downward vertical**, as in the diagram below.



(a) The particle is projected horizontally from the lowest point of the vertical circle with speed $2\sqrt{gr}$, where g is the magnitude of the acceleration due to gravity.
At the instant the string becomes slack, calculate:
(i) the value of θ; **6**
(ii) the speed of P, given that $r = 0{\cdot}4$ metres. **2**

(b) With the same value of r, calculate the increase in the horizontal speed at the lowest point which would.be required for the particle to execute a complete circle. **5**

(c) What is the maximum initial horizontal speed which could be given to the particle at the lowest point of the same circle for the particle never to rise above the horizontal through O? **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)(i)	6	2.4		6	2.4.11	2.4.6/10,2.1.3	2000 SY5 Q8
(a)(ii)	2	2.4	2				
(b)	5	2.4	1	4	2.4.11	2.4.6/10,2.1.3	
(c)	2	2.4	2				

(a) (i)

$$T - mg\cos\theta = \frac{mv^2}{r} \qquad \mathbf{1}$$

$$\frac{1}{2}mv^2 + mgh = \frac{1}{2}mu^2 \qquad \mathbf{1}$$

$$\tfrac{1}{2}mv^2 + mgr(1 - \cos\theta) = \tfrac{1}{2}mu^2$$

$$v^2 + 2gr(1 - \cos\theta) = 4gr$$

$$v^2 = 2gr(1 + \cos\theta) \qquad \mathbf{1}$$

$$\frac{mv^2}{r} = 2mg(1 + \cos\theta)$$

$$T = mg\cos\theta + \frac{mv^2}{r} \qquad \mathbf{1}$$

$$= mg\cos\theta + 2mg(1 + \cos\theta)$$

$$= mg(2 + 3\cos\theta) \qquad \mathbf{1}$$

when $T = 0$

$$2 + 3\cos\theta = 0$$

$$\cos\theta = -\frac{2}{3}$$

$$\theta \approx 132° \qquad \mathbf{1}$$

(ii)

$$v^2 = 2gr(1 + \cos\theta)$$

$$= 2gr(1 - \frac{2}{3}) \qquad \mathbf{1}$$

$$= \frac{2gr}{3}$$

$$v \approx 1{\cdot}62 \text{ m s}^{-1} \qquad \mathbf{1}$$

(b)

$$v^2 + 2gr(1 - \cos\theta) = u^2$$

$$v^2 = u^2 - 2gr(1 - \cos\theta) \qquad \mathbf{1}$$

$$T = mg\cos\theta + \frac{mv^2}{r}$$

$$T = mg\cos\theta + \frac{m}{r}\left(u^2 - 2gr(1 - \cos\theta)\right)$$

$$= mg\cos\theta + \frac{mu^2}{r} - 2mg(1 - \cos\theta)$$

$$= \frac{mu^2}{r} + mg\cos\theta - 2mg + 2mg\cos\theta$$

$$= \frac{mu^2}{r} + mg(3\cos\theta - 2) \qquad \mathbf{1}$$

$$T \geqslant 0 \text{ for } 0° \leqslant \theta \leqslant 180°$$

$$\frac{mu^2}{r} + mg(3\cos\theta - 2) \geqslant 0 \qquad \mathbf{1}$$

$$u^2 \geqslant gr(2 - 3\cos\theta)$$

$$u^2 \geqslant gr(2 - 3 \times (-1)) = 5gr \qquad \mathbf{1}$$

$$\text{Increase in speed} = \sqrt{5gr} - \sqrt{4gr}$$

$$\approx \sqrt{19{\cdot}6} - \sqrt{15.68}$$

$$\approx 4{\cdot}43 - 3{\cdot}96 = 0{\cdot}47 \text{ m s}^{-1} \qquad \mathbf{1}$$

(c) We want $T = 0$ when $\theta = \frac{\pi}{2}$

$$T = \frac{mu^2}{r} + mg(3\cos\theta - 2)$$

$$0 = \frac{mu^2}{r} + mg(0 - 2) \qquad \mathbf{1}$$

$$0 = \frac{u^2}{r} + 2g$$

$$u^2 = 2gr$$

$$u = \sqrt{2gr} \approx \sqrt{7{\cdot}84} \approx 2.8 \text{ m s}^{-1} \qquad \mathbf{1}$$

A feather of mass 0·008 kg and a hammer of mass 1 kg are dropped from rest at a height of 10 m above a horizontal surface. Both the feather and the hammer can be modelled as particles falling vertically under constant gravity. The air resistance acting against the feather has magnitude $gv^2/500$ newtons and the air resistance acting against the hammer has magnitude $gv^2/4000$newtons, where v m s^{-1} amd V m s^{-1} are the speeds of the feather and hammer respectively after time t seconds, and g m s^{-2} is the magnitude of the acceleration due to gravity.

(a) Calculate the terminal speed of the feather and find the time taken, from the start of the motion, for the feather to reach 99% of its terminal speed. **8**

[You may assume that $\frac{4}{4 - v^2} = \frac{1}{2 + v} + \frac{1}{2 - v}$.]

(b) Calculate the speed of the hammer when it hits the horizontal surface. **7**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	8	2.5	3	5	2.5.2/3	2.5.5	2000 SY5 Q9
(b)	7	2.5	1	6	2.5.2/3		

(a)

$$ma = mg - \frac{gv^2}{500}$$

$$0{\cdot}008a = 0{\cdot}008g - \frac{gv^2}{500} \qquad \mathbf{1}$$

$$a = g - \frac{gv^2}{4}$$

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{4}\right) \qquad \mathbf{1}$$

Terminal speed when $a = 0 \Rightarrow \frac{v^2}{4} = 1$ **1**

$$v = 2 \text{ m s}^{-1} \qquad \mathbf{1}$$

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{4}\right)$$

$$\int \frac{4\,dv}{4 - v^2} = \int g\,dt \qquad \mathbf{1}$$

$$\int \left(\frac{1}{2 + v} + \frac{1}{2 - v}\right) dv = \int g\,dt$$

$$\ln|2 + v| - \ln|2 - v| = gt + c \qquad \mathbf{1}$$

when $t = 0, v = 0 \Rightarrow c = 0$

$$t = \frac{1}{g}\ln\left|\frac{2 + v}{2 - v}\right| \qquad \mathbf{1}$$

when $v = 0{\cdot}99 \times 2 = 1{\cdot}98$

$$t = \frac{1}{g}\ln\left|\frac{3{\cdot}98}{0{\cdot}02}\right| \approx 0{\cdot}54 \text{ s} \qquad \mathbf{1}$$

(b)

$$ma = mg - \frac{gV^2}{4000}$$

$$a = g\left(1 - \frac{V^2}{4000}\right)$$

$$V\frac{dV}{dx} = g\left(1 - \frac{V^2}{4000}\right) \qquad \mathbf{1}$$

$$\int \frac{V}{1 - \frac{V^2}{4000}}\,dV = \int g\,dx \qquad \mathbf{1}$$

$$-2000\ln\left|1 - \frac{V^2}{4000}\right| = gx + c \qquad \mathbf{1}$$

when $x = 0, v = 0 \Rightarrow c = 0$

$$x = \frac{-2000}{g}\ln\left|1 - \frac{V^2}{4000}\right| \qquad \mathbf{1}$$

when $x = 10$

$$10 = \frac{-2000}{g}\ln\left|1 - \frac{V^2}{4000}\right|$$

$$e^{-g/200} = 1 - \frac{V^2}{4000} \qquad \mathbf{1}$$

$$V^2 = 4000\left(1 - e^{-g/200}\right) \qquad \mathbf{1}$$

$$V = \sqrt{4000\left(1 - e^{-9{\cdot}8/200}\right)}$$

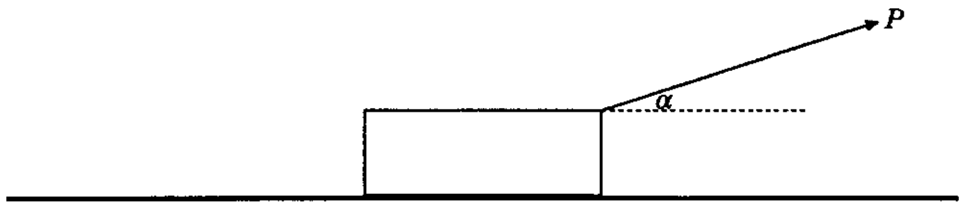
$$\approx 13{\cdot}8 \text{ m s}^{-1} \qquad \mathbf{1}$$

The coefficient of friction between a body, of mass m, and a particular plane surface is μ.

(a) When the plane surface is inclined at the angle of friction λ to the horizontal the body is just on the point of slipping down the plane. Show that

$$\mu = \tan\lambda. \qquad 4$$

(b) When the plane surface is horizontal, a light inextensible tow rope is attached to one end of the body. A pulling force of magnitude P is applied to the tow rope which is inclined at angle α to the horizontal, as in the diagram below.



(i) Given that the body is just on the point of slipping across the plane, show that

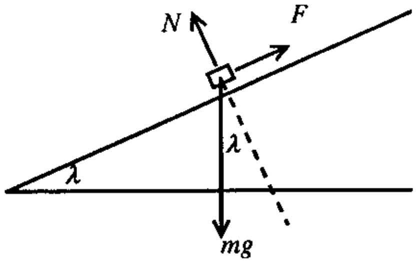
$$P = \frac{\mu m g}{\cos\alpha + \mu\sin\alpha},$$

where g is the magnitude of the acceleration due to gravity. **3**

(ii) Show that the minimum value of P that can pull the body across the plane occurs when the angle ais equal to the angle of friction λ and calculate this minimum value of P when $\mu = 0{\cdot}4$ and $m = 50$ kg. **8**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	1.4	4		1.4.6	1.4.3	2000 SY5 Q10
(b)(i)	3	1.4		3	1.4.6	1.4.3	
(b)(ii)	8	1.4	1	7	1.4.11	1.4.5	

(a)



Resolving perpendicular to plane

$$N = mg\cos\lambda \qquad 1$$

Resolving parallel to plane

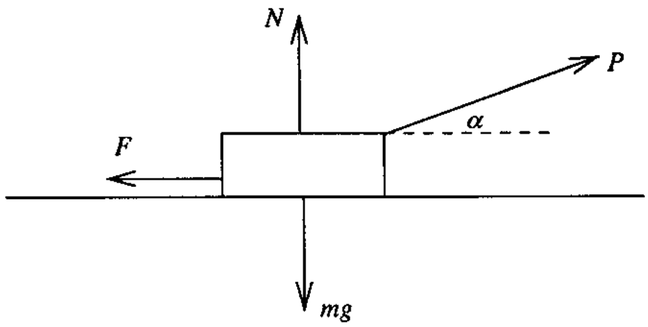
$$F = mg\sin\lambda \qquad 1$$

$$\mu N = mg\sin\lambda \qquad 1$$

$$\frac{\mu N}{N} = \frac{mg\sin\lambda}{mg\cos\lambda}$$

$$\mu = \tan\lambda \qquad 1$$

(b) (i)



$$N + P\sin\alpha = mg \qquad F = P\cos\alpha \qquad 1$$

$$N = mg - P\sin\alpha \qquad \mu N = P\cos\alpha \qquad 1$$

$$\mu mg - \mu P\sin\alpha = P\cos\alpha$$

$$P = \frac{\mu mg}{\cos\alpha + \mu\sin\alpha} \qquad 1$$

(ii)

$$P(\alpha) = \frac{\mu mg}{\cos\alpha + \mu\sin\alpha}$$

$$P'(\alpha) = \frac{\mu mg(\sin\alpha - \mu\cos\alpha)}{(\cos\alpha + \mu\sin\alpha)^2} \qquad 2$$

$$P'(\alpha) = 0 \Rightarrow \sin\alpha - \mu\cos\alpha = 0$$

$$\mu = \tan\alpha \qquad 1$$

but $\mu = \tan\lambda \Rightarrow \tan\alpha = \tan\lambda \Rightarrow \alpha = \lambda$ 1

Verification of nature of stationary value 1

$$P_{\min} = \frac{\mu mg}{\cos\lambda + \mu\sin\lambda}$$

$$= \frac{\mu mg}{\frac{1}{\sqrt{1+\mu^2}} + \frac{\mu^2}{\sqrt{1+\mu^2}}}$$

$$= \frac{\mu mg}{\sqrt{1+\mu^2}} \qquad 1$$

$$= \frac{0{\cdot}4 \times 50 \times 9{\cdot}8}{\sqrt{1 + 0{\cdot}4^2}} \qquad 1$$

$$= 182\text{ N} \qquad 1$$