SCOTTISH QUALIFICATIONS AUTHORITY

2004 Applied Mathematics

Advanced Higher – Section B

Finalised Marking Instructions

Advanced Higher Applied Mathematics 2004
Solutions for Section B (Numerical Analysis 1 and 2)

B1. $f(x) = \ln(2-x) \qquad f'(x) = \dfrac{-1}{(2-x)} \qquad f''(x) = \dfrac{-1}{(2-x)^2} \qquad f'''(x) = \dfrac{-2}{(2-x)^3}$ **1**

Taylor polynomial is

$$p(1+h) = \ln 1 - h - \frac{h^2}{2} - \frac{2h^3}{6}$$

$$= -h - \frac{h^2}{2} - \frac{h^3}{3}$$ **1**

For $\ln 1{\cdot}1$, $h = -0{\cdot}1$ and $p(0{\cdot}9) = 0{\cdot}1 - 0{\cdot}005 + 0{\cdot}00033 = 0{\cdot}0953$. **1,1**

$$p(a+h) = \ln(2-a) - \frac{1}{2-a}h$$ **1**

Hence expect $f(x)$ to be more sensitive in I_2 since coefficient of h is much larger. **1**

B2. $L(2{\cdot}5)$

$$= \frac{(2{\cdot}5-1{\cdot}5)(2{\cdot}5-3{\cdot}0)(2{\cdot}5-4{\cdot}5)}{(0{\cdot}5-1{\cdot}5)(0{\cdot}5-3{\cdot}0)(0{\cdot}5-4{\cdot}5)}1{\cdot}737 + \frac{(2{\cdot}5-0{\cdot}5)(2{\cdot}5-3{\cdot}0)(2{\cdot}5-4{\cdot}5)}{(1{\cdot}5-0{\cdot}5)(1{\cdot}5-3{\cdot}0)(1{\cdot}5-4{\cdot}5)}2{\cdot}412$$

$$+ \frac{(2{\cdot}5-0{\cdot}5)(2{\cdot}5-1{\cdot}5)(2{\cdot}5-4{\cdot}5)}{(3{\cdot}0-0{\cdot}5)(3{\cdot}0-1{\cdot}5)(3{\cdot}0-4{\cdot}5)}3{\cdot}284 + \frac{(2{\cdot}5-0{\cdot}5)(2{\cdot}5-1{\cdot}5)(2{\cdot}5-3{\cdot}0)}{(4{\cdot}5-0{\cdot}5)(4{\cdot}5-1{\cdot}5)(4{\cdot}5-3{\cdot}0)}2{\cdot}797$$ **2**

$$= -\frac{1 \times 1{\cdot}737}{10} + \frac{2 \times 2{\cdot}412}{4{\cdot}5} + \frac{4 \times 3{\cdot}284}{2{\cdot}5 \times 2{\cdot}25} - \frac{1 \times 2{\cdot}797}{18}$$

$$= -0{\cdot}1737 + 1{\cdot}0720 + 2{\cdot}3353 - 0{\cdot}1554 = 3{\cdot}078$$ **2**

B3. $\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0) = f_2 - 2f_1 + f_0$ **1**

Maximum rounding error $= \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon$. **1**

$\Delta^2 f_0 = 2{\cdot}618 - 2 \times 2{\cdot}369 + 2{\cdot}124 = 0{\cdot}004$ **1**
and $4\varepsilon = 4 \times 0{\cdot}0005 = 0{\cdot}002$. **1**

$\Delta^2 f_0$ appears to be significantly different from 0. **1**

B4. (a) Difference table is:

i	x	$f(x)$	diff1	diff2	diff3
0	0	1·023	352	–95	3
1	0·5	1·375	257	–92	–4
2	1	1·632	165	–96	
3	1·5	1·797	69		
4	2	1·866			

2

(b) $p = 0{\cdot}3$

$$f(0{\cdot}65) = 1{\cdot}375 + 0{\cdot}3(0{\cdot}257) + \frac{(0{\cdot}3)(-0{\cdot}7)}{2}(-0{\cdot}092)$$ **2**

$$= 1{\cdot}375 + 0{\cdot}077 + 0{\cdot}010 = 1{\cdot}462$$ **1**

(or, with $p = 1{\cdot}3$, $1{\cdot}023 + 0{\cdot}458 - 0{\cdot}019$).

B5. $f(x) = (((x - 1{\cdot}1)x + 1{\cdot}7)x)x - 3{\cdot}2$ and $f(1{\cdot}3) = 0{\cdot}1124$. **1,1**
Since f is positive and increasing at $x = 1{\cdot}3$, root appears to occur for $x < 1{\cdot}3$. **1**
$f(x)_{\min} = (((x - 1{\cdot}15)x + 1{\cdot}65)x)x - 3{\cdot}25$
$f(1{\cdot}3)_{\min} = -0{\cdot}132$ (opposite sign), so root may occur for $x > 1{\cdot}3$. **1,1**

B6. In diagonally dominant form,

$$\begin{aligned} 4x_1 - 0.3x_2 + 0.5x_3 &= 6.1 \\ 0.5x_1 - 7x_2 + 0.7x_3 &= 3.7 \\ 0.3x_1 + 2x_3 &= 8.6. \end{aligned}$$ **1**

The diagonal coefficients of x are large relative to the others, so system is likely to be stable. (Or, this implies equations are highly linearly independent, or, determinant of system is large.) **1**

Rewritten equations are:

$$\begin{aligned} x_1 &= (6.1 + 0.3x_2 - 0.5x_3)/4 \\ x_2 &= (-3.7 + 0.5x_2 + 0.7x_3)/7 \\ x_3 &= (8.6 - 0.3x_1)/2 \end{aligned}$$

Gauss Seidel table is:

x_1	x_2	x_3
0	0	0
1·525	–0·420	4·071
0·985	–0·051	4·152
1·002	–0·042	4·150
1·003	–0·042	

Hence (2 decimal places) $x_1 = 1{\cdot}00$; $x_2 = -0{\cdot}04$; $x_3 = 4{\cdot}15$. **4**

B7.

Tableau is: $\begin{pmatrix} 2{\cdot}6 & 0 & 1{\cdot}622 & 0{\cdot}742 & 0{\cdot}479 & 0 \\ 0 & 6{\cdot}469 & 1{\cdot}923 & -0{\cdot}538 & 1 & 0 \\ 0 & 0 & 3{\cdot}604 & -0{\cdot}415 & 0{\cdot}128 & 1 \end{pmatrix}$

$\sim \begin{pmatrix} 2{\cdot}6 & 0 & 0 & 0{\cdot}929 & 0{\cdot}421 & -0{\cdot}450 \\ 0 & 6{\cdot}469 & 0 & -0{\cdot}317 & 0{\cdot}932 & -0{\cdot}534 \\ 0 & 0 & 3{\cdot}604 & -0{\cdot}415 & 0{\cdot}128 & 1 \end{pmatrix}$ $(R_1 - 1{\cdot}622R_3/3{\cdot}604)$ $(R_2 - 1{\cdot}923R_3/3{\cdot}604)$ **3**

$\sim \begin{pmatrix} 1 & 0 & 0 & 0{\cdot}357 & 0{\cdot}162 & -0{\cdot}173 \\ 0 & 1 & 0 & -0{\cdot}049 & 0{\cdot}144 & -0{\cdot}083 \\ 0 & 0 & 1 & -0{\cdot}115 & 0{\cdot}036 & 0{\cdot}277 \end{pmatrix}$ (dividing by diagonal elements) **1**

Hence $\mathbf{A}^{-1} = \begin{pmatrix} 0{\cdot}36 & 0{\cdot}16 & -0{\cdot}17 \\ -0{\cdot}05 & 0{\cdot}14 & -0{\cdot}08 \\ -0{\cdot}12 & 0{\cdot}04 & 0{\cdot}28 \end{pmatrix}$. **1**

accuracy **1**

B8. (a)

x	y	$f(x, y)$	$hf(x, y)$	
1	1	0·414	0·041	
1·1	1·041	0·514	0·051	**2**
1·2	1·092	0·620	0·062	
1·3	1·154			**1**

Global truncation error is first order. **1**

(b) Predictor-corrector calculation (with one corrector application) is:

x	y	$y' = \sqrt{x^2 + 2y - 1} - 1$	y_P	y_P'	$\frac{1}{2}h(y' + y_P')$	
1	1	0·4142	1·0414	0·5142	0·0464	
1·1	1·0464					**4**

The difference in the (rounded) second decimal place between the values of $x(1{\cdot}1)$ in the two calculations suggests that the second decimal place cannot be relied upon in the first calculation. **1**

B9. Trapezium rule calculation is:

x	$f(x)$	m	$mf_1(x)$	$mf_2(x)$
1	1·2690	1	1·2690	1·2690
1·25	1·1803	2		2·3606
1·5	0·9867	2	1·9734	1·9734
1·75	0·6839	2		1·3678
2	0·2749	1	0·2749	0·2749
			3·5173	7·2457

2

Hence $I_1 = 3{\cdot}5173 \times 0{\cdot}5/2 = 0{\cdot}8793$ and $I_2 = 7{\cdot}2457 \times 0{\cdot}25/2 = 0{\cdot}9057$. 1

Difference table is:

−887	−1049
−1936	−1092
−3028	−1062
−4090	

2

$|\text{max truncation error}| = 1 \times 0{\cdot}1092/12 \approx 0{\cdot}009$ 1

Hence $I_2 = 0{\cdot}91$ or $0{\cdot}9$. 1

Expect to reduce error by factor 4. 1

With n strips and step size $2h$, Taylor series for expansion of an integral I approximated by the trapezium rule is:

$I = I_n + C(2h)^2 + D(2h)^4 + \ldots = I_n + 4Ch^2 + 16Dh^4 + \ldots$ (a)

With $2n$ strips and step size h, we have: $I = I_{2n} + Ch^2 + Dh^4 + \ldots$ (b) 2

4(b) – (a) gives $3I = 4I_{2n} - I_n - 12Dh^4 + \ldots$

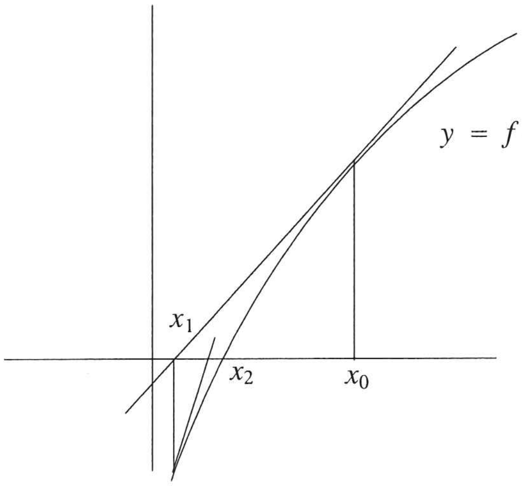
i.e. $I \approx (4I_{2n} - I_n)/3 = I_{2n} + (I_{2n} - I_n)/3$ 1

$I_R = (4 \times 0{\cdot}9057 - 0{\cdot}8793)/3 = 0{\cdot}914$ 1

B10. Gradient of $y = f(x)$ at x_0 is $f'(x_0) = \dfrac{f(x_0)}{x_0 - x_1}$. **1**

Hence $x_1 - x_0 = -\dfrac{f(x_0)}{f'(x_0)}$, i.e. $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

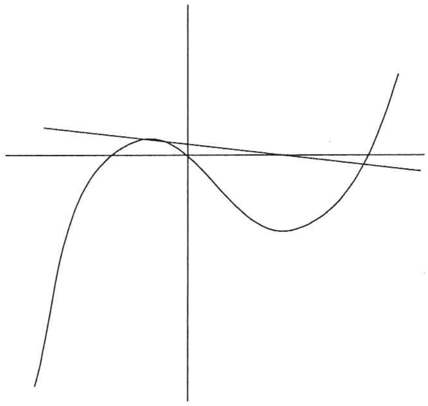
Likewise $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ and in general $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. **1**



1

$f(x) = e^{-x} + x^4 - 2x^3 - 5x^2 - 1$ and $f'(x) = -e^{-x} + 4x^3 - 6x^2 - 10x$; $x_0 = 3{\cdot}5$ **1**

Root is 3·47 (2 decimal places). **1**



In a situation such as diagrammed, the Newton-Raphson method depends for convergence on the point of intersection of tangent with x-axis being closer to the root than the initial point. In the interval $[-0{\cdot}3,\ 0]$ there must be a TV of $f(x)$ so that $f'(x) = 0$ and the point of intersection may be far from initial point; so iteration may lead to a different root. **2**

For bisection, $f(-1{\cdot}1) = 0{\cdot}080$; $f(-1) = -0{\cdot}281$

$f(-1{\cdot}05) = -0{\cdot}124$

$f(-1{\cdot}075) = -0{\cdot}208$; **1**

$f(-1{\cdot}0875) = 0{\cdot}024$ **1**

Hence root lies in $[-1{\cdot}0875. - 1{\cdot}075]$. **1**