Marks

Section B (Numerical Analysis 1 and 2)

ONLY candidates doing the course Numerical Analysis 1 and 2 and one unit chosen from Mathematics 1 (Section D), Statistics 1 (Section E) and Mechanics 1 (Section G) should attempt this Section.

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

B1. The polynomial p is the Taylor polynomial of degree three for a function f near $x = 1$.

Given the function $f(x) = \ln(2 - x)$, where $x < 2$, express $p(1 + h)$ in the form $c_0 + c_1h + c_2h^2 + c_3h^3$. **2**

Estimate the value of $\ln(1{\cdot}1)$ using this approximation, giving your answer to four decimal places. **2**

Write down the first degree Taylor polynomial for f near $x = a$ where $a < 2$. Given the intervals $I_1[0{\cdot}1,\ 0{\cdot}2]$ and $I_2[1{\cdot}8,\ 1{\cdot}9]$, in which interval would you expect $f(x)$ to be more sensitive to small changes in x? **2**

B2. The following data are available for a function f:

x	0·5	1·5	3·0	4·5
$f(x)$	1·737	2·412	3·284	2·797

Use the cubic Lagrange interpolation formula to estimate $f(2{\cdot}5)$. **4**

B3. In the usual notation for forward differences of function values $f(x)$ tabulated at equally spaced values of x,

$$\Delta f_i = f_{i+1} - f_i$$

where $f_i = f(x_i)$ and $i = \dots, -2, -1, 0, 1, 2, \dots$.

Show that $\Delta^2 f_0 = f_2 - 2f_1 + f_0$. **1**

If each value of f_i is subject to an error whose magnitude is less than or equal to ε, determine the magnitude of the maximum possible rounding error in $\Delta^2 f_0$. **1**

Rounded values of a function f are known to be $f_0 = 2{\cdot}124$, $f_1 = 2{\cdot}369$, $f_2 = 2{\cdot}618$. Obtain $\Delta^2 f_0$ and the magnitude of the maximum rounding error in $\Delta^2 f_0$. **2**

Hence state whether or not this second difference appears to be significantly different from zero. **1**

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B4. The following data (accurate to the degree implied) are available for a function f:

x	0	0·5	1	1·5	2
$f(x)$	1·023	1·375	1·632	1·797	1·866

(*a*) Construct a difference table of third order for the data. **2**

(*b*) Using the Newton forward difference formula of degree two, and working to three decimal places, obtain an approximation to $f(0·65)$. **3**

B5. Express the polynomial $f(x) = x^4 - 1·1x^3 + 1·7x^2 - 3·2$ in nested form and evaluate $f(1·3)$. **2**

It is known that the equation $f(x) = 0$ has a root very close to $x = 1·3$ and that $f(x)$ is increasing for $1 < x < 2$. State whether the root appears to occur for $x < 1·3$ or for $x > 1·3$. **1**

Given that the term in x^4 is exact and that the other coefficients of $f(x)$ are rounded to the accuracy implied, show by considering the minimum possible value of $f(1·3)$ that it is possible that the root may in fact be located on the other side of the point $x = 1·3$. **2**

B6. Write the equations

$$\begin{aligned} 0·3x_1 \qquad\qquad + \quad 2x_3 &= 8·6 \\ 4x_1 - 0·3x_2 + 0·5x_3 &= 6·1 \\ 0·5x_1 - \quad 7x_2 + 0·7x_3 &= 3·7, \end{aligned}$$

in diagonally dominant form. Give a reason for stating that the equations are not ill-conditioned. **2**

Use the Gauss-Seidel iterative procedure with $x_1 = x_2 = x_3 = 0$ as a first approximation to solve the equations correct to two decimal places. **4**

B7. In the calculation using Gaussian elimination to obtain the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 2·6 & -3·1 & 0·7 \\ 1·4 & 4·8 & 2·3 \\ 0·9 & -1·9 & 3·6 \end{pmatrix}$$

with the diagonalisation process carried out to the extent shown, the tableau of elements is:

$$\begin{pmatrix} 2·6 & 0 & 1·622 & 0·742 & 0·479 & 0 \\ 0 & 6·469 & 1·923 & -0·538 & 1 & 0 \\ 0 & 0 & 3·604 & -0·415 & 0·128 & 1 \end{pmatrix}.$$

Continuing to work to the same accuracy, complete the determination of the inverse of $\mathbf{A}$, giving your answer with elements rounded to two decimal places. **6**

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B8. The differential equation $\frac{dy}{dx} = \sqrt{x^2 + 2y - 1} - 1$ with $y(1) = 1$ is to be solved numerically.

(*a*) Use Euler's method with a step length of 0·1 to obtain an approximation at $x = 1{\cdot}3$ to the solution of this equation. Perform the calculations using three decimal place accuracy. **3**

State the order of the truncation order in this solution. **1**

(*b*) Use the predictor-corrector method with Euler's method as predictor and the trapezium rule as corrector to obtain a solution of this equation at $x = 1{\cdot}1$. Use one application of the corrector with a step length $h = 0{\cdot}1$ and perform the calculation using four decimal place accuracy. **4**

Comment on what this answer reveals about the accuracy of the estimate of $y(1{\cdot}1)$ obtained in part (*a*) of the question. **1**

B9. The following data are available for a function f:

x	1	1·25	1·5	1·75	2
$f(x)$	1·2690	1·1803	0·9867	0·6839	0·2749

Use the composite trapezium rule with two strips and four strips to obtain estimates I_1 and I_2 respectively for the integral $I = \int_1^2 f(x)\,dx$.

Perform the calculations using four decimal places. **3**

By constructing an appropriate difference table, obtain an estimate of the maximum truncation error in I_2. **3**

Hence state the value of I_2 to a suitable accuracy. **1**

If data of similar accuracy were available for the intermediate points with $x = 1{\cdot}125$, $1{\cdot}375$, etc, and the calculations were done with eight strips, by what factor would you expect the truncation error to be reduced? **1**

By considering appropriate Taylor series expansions for a definite integral, establish Richardson's formula to improve the accuracy of the trapezium rule by interval halving. **3**

Use Richardson extrapolation to obtain an improved estimate I_3 for I based on the values of I_1 and I_2. **1**

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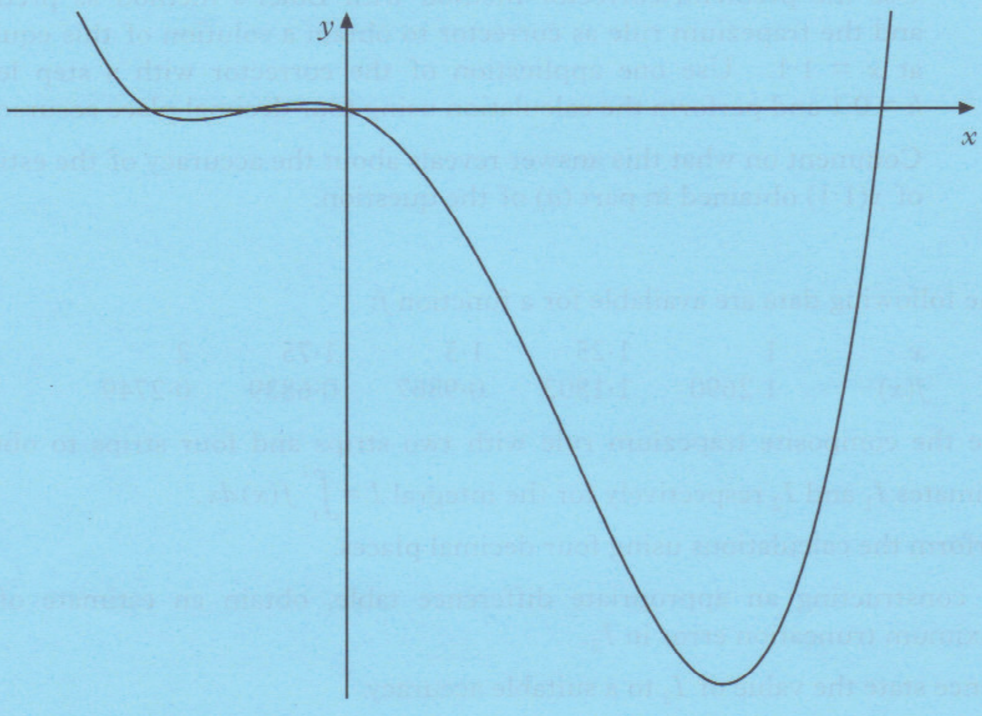
B10. The equation $f(x) = 0$ has a root close to $x = x_0$. By drawing a suitable graph to illustrate this situation, derive the formula for the first iteration of the Newton-Raphson method of solution $f(x) = 0$. Hence explain how the general formula is obtained. **3**

The diagram shows part of the graph of $f(x) = e^{-x} + x^4 - 2x^3 - 5x^2 - 1$ and shows that $f(x) = 0$ has four distinct real roots.



One root is known to lie in the interval [3·4, 3·6]. Use the Newton-Raphson method to determine this root correct to two decimal places. **2**

The equation has a root at $x = 0$ and another in the interval [–0·3, 0]. Use a diagram to explain why the Newton-Raphson method may be difficult to use to determine this negative root. **2**

The fourth root is given to lie in the interval [–1·1, –1]. Use three applications of the bisection method to determine a more accurate estimate of the interval in which this root lies. **3**

[*END OF SECTION B*]

Candidates who have attempted Section B (Numerical Analysis 1 and 2) should now attempt ONE of the following

Section D (Mathematics 1) on *Page fifteen*

Section E (Statistics 1) on *Pages sixteen* and *seventeen*

Section G (Mechanics 1) on *Pages twenty* and *twenty-one*.