SCOTTISH QUALIFICATIONS AUTHORITY

Principal Assessor Report 2002

Assessment Panel:	**Mathematics**

Qualification area

Subject(s) and Level(s) included in this report	**MATHEMATICS (Advanced Higher)**

Statistical information: update

Number of entries in 2001	
Pre appeal	1126
Post appeal	

Number of entries in 2002	
Pre appeal	2499
Post appeal	

General comments re entry numbers

The increase reflects the demise of CSYS mathematics. However, the figure of 2499 is, higher than the CSYS General Paper was at any stage.
The approximate numbers involved in the options are:
A and B: 2210
A and C: 115
A and D: 70
A and E: 110

General comments

The paper set in 2001 was straightforward. The 2002 paper was designed to be a bit harder but the standards achieved in it were disappointing. This may have been because of the large increase in centres offering Advanced Higher for the first time.

Comments on any significant changes in percentages or distribution of awards

The proportions of A, B, C awards are broadly in line with previous CSYS profiles. It is to be hoped that the proportion of awards will increase as teachers become more familiar with the course.

Grade boundaries at C, B and A for each subject area included in the report

Advanced Higher Mathematics: Pass mark stage

Maximum mark 100		
A	**B**	**C**
66%	53%	41%

General commentary on grade boundaries

Notional percentage cut-offs for each grade

Question papers and their associated marking schemes are designed to be of the required standard and to meet the assessment specification for the subject/level concerned.
For National courses the examination paper(s) are set in order that a score of approximately 50% of the total marks for all components merits a grade C (based on the grade descriptions for that grade), and similarly a score of 70 % for a grade A. The lowest mark for a grade B is set by the computer software as half way between the C and A grade boundaries.

Comments on grade boundaries for each subject area

The cut-offs are lower than we would have liked. This is probably due to a combination of a more challenging paper and also candidates whose preparation seemed to be poor.

Comments on candidate performance

General comments

Overall, it was disappointing to see that candidates were able to cope less well that we had hoped. Only the first question was found to be straight forward. The rest all provided challenges to a greater or lesser extent. The performances in the Options were broadly similar, although those few who took Numerical Analysis scored slightly better.

Areas of external assessment in which candidates performed well

In section A, the only topic which was uniformly done well was the Gaussian elimination, the first derivative in question 4 was not far behind.

Areas of external assessment in which candidates had difficulty

There was a disappointingly high number of candidates who made careless errors. Errors in copying down questions and their own working were noticeable – perhaps the often poor writing and badly set out answers contributed to this problem.
Section A
Question 2, on the roots of a polynomial, should have been routine but, sadly, was not. Question 8, on curve sketching, should also have been routine but the setter's attempt to encourage candidates to use a partial fraction approach proved to be a failure.
Most of the other questions, through intrinsic difficulty or the newness of the topic, caused problems.
Question 10 was rarely attacked in any robust way.
Many candidates failed to obtain the relatively easy marks which were often available at the start of a question.
Section B also caused more difficulty than anticipated. Whilst a large number of candidates performed well, many showed evidence of lack of knowledge and preparation.

Areas of common misunderstanding

None were apparent.

Recommendations

Feedback to centres

Naturally, as centres become more familiar with the course, preparation and performance will improve.

It will also be advantageous for candidates to improve their examination technique. Particular point are

- Make sure writing is clear.
- Make sure work is clearly set out.
- Take particular care over manipulation.
- Look for accessible marks at the start of a question or one of its parts.

2002 Mathematics

Advanced Higher

Finalised Marking Instructions

SECTION A (Mathematics 1 and 2)

A1.

$$\begin{array}{cccc} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 5 & 5 \end{array} \Rightarrow \begin{array}{cccc} 1 & 1 & 3 & 2 \\ & -1 & -5 & -2 \\ & -1 & -4 & -1 \end{array}$$

$$\Rightarrow \begin{array}{cccc} 1 & 1 & 3 & 2 \\ & -1 & -5 & -2 \\ & & -1 & -1 \end{array}$$

$$z = 1; \quad y = -3; \quad x = 2$$

Second row 1 mark
Third row 1 mark

Third row 1 mark

Values 2E1.
(available whatever method used above)

Total 5

A2.

$$i^4 + 4i^3 + 3i^2 + 4i + 2$$

$$= 1 - 4i - 3 + 4i + 2 = 0$$

Since i is a root, $-i$ must also be a root. Thus factors $(z - i)$ and $(z + i)$ give a quadratic factor $z^2 + 1$.

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad z^2 + 4z + 2 \\ z^2 + 1 & z^4 + 4z^3 + 3z^2 + 4z + 2 \\ & \underline{z^4 \quad\quad\quad + z^2} \\ & \quad\quad 4z^3 + 2z^2 + 4z \\ & \quad\quad \underline{4z^3 \quad\quad\quad + 4z} \\ & \quad\quad\quad\quad 2z^2 \quad\quad\quad + 2 \end{array}$$

Solving $z^2 + 4z + 2 = 0$ gives

$$z = -2 \pm \sqrt{2}.$$

1 mark for verifying and stating

1 for getting $-i$.

1 for $z^2 + 1$ is a factor.

1 for factorisation.

1 for the other two roots.

Total 5

A3.

At A, $x = -1$ so $t^2 + t - 1 = -1$ giving $t = 0$ or $t = -1$. When $t = 0$, $y = 2$. When $t = -1$, $y = 5$ so A is on the curve.

$$\frac{dx}{dt} = 2t + 1; \ \frac{dy}{dt} = 4t - 1$$

$$\frac{dy}{dx} = \frac{4t - 1}{2t + 1}.$$

When $t = -1$, $\frac{dy}{dx} = \frac{-5}{-1} = 5$.

The equation is

$$(y - 5) = 5(x + 1)$$

$$y = 5x + 10$$

1 for solving a quadratic.

1 for the other coordinate.

1 for $\frac{dx}{dt}$ **and** $\frac{dy}{dt}$.

1 for $\frac{dy}{dx}$.

1 for the gradient is 5.

1 for an equation.

Total 6

A4.

(a) $f(x) = \sqrt{x}e^{-x} = x^{1/2}e^{-x}$

$f'(x) = \tfrac{1}{2}x^{-1/2}e^{-x} + x^{1/2}(-1)e^{-x}$

$= \dfrac{1}{2\sqrt{x}} e^{-x}(1 - 2x)$

1 method mark — 1 for first term; 1 for second term

1 for a factorised form

(b) $y = (x + 1)^2 (x + 2)^{-4}$

$\log y = 2\log(x+1) - 4\log(x+2)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2}{x+1} - \dfrac{4}{x+2}$

$\dfrac{dy}{dx} = \left(\dfrac{2}{x+1} - \dfrac{4}{x+2}\right) y$

$a = 2;\ b = -4$

1 for taking logs and expanding

1 for differentiating

A logarithmic approach is needed.

1 for rearranging

Total 7

A5.

$\int_0^1 \ln(1 + x)\, dx$

$= \int_0^1 \ln(1 + x)\ .\ 1\ dx$

$= \left[x \ln(1 + x) - \int \dfrac{1}{1 + x}.x\, dx\right]_0^1$

$= \left[x \ln(1 + x) - \int \left(1 - \dfrac{1}{1 + x}\right) dx\right]_0^1$

$= [x \ln(1 + x) - x + \ln(1 + x)]_0^1$

$= [\ln 2 - 1 + \ln 2] - [0 - 0 + 0]$

$= 2 \ln 2 - 1\ [\approx 0.3863]$.

1 for introducing the factor of 1

1 for second term

2 marks for correct manipulation and integration of the second term

1 for limits

Total 5

A6.

$x + 2 = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$

1 for derivative

Also, $x = 2 \tan\theta - 2$, so

$x^2 = 4 \tan^2\theta - 8 \tan\theta + 4$, giving

$x^2 + 4x + 8 = 4 \tan^2\theta + 4$

1 for manipulation

$\int \dfrac{dx}{x^2 + 4x + 8} = \int \dfrac{2 \sec^2\theta\, d\theta}{4(\tan^2\theta + 1)}$

1 for substitution

$= \dfrac{1}{2}\int \dfrac{\sec^2\theta\, d\theta}{\tan^2\theta + 1}$

$= \dfrac{1}{2}\int d\theta$

1 for simplifying

$= \dfrac{1}{2}\theta + c$

$= \dfrac{1}{2} \tan^{-1}\left(\dfrac{x + 2}{2}\right) + c$

1 for finishing

Total 5

A7.

When $n = 1$, $4^n - 1 = 4 - 1 = 3$ so true when $n = 1$.

Assume $4^k - 1$ is divisible by 3.

Consider $4^{k+1} - 1$.

$$4^{k+1} - 1 = 4.4^k - 1$$
$$= (3 + 1)4^k - 1$$
$$= 3.4^k + \left(4^k - 1\right)$$

Since both terms are divisble by 3 the result is true for $k + 1$.

Thus since true for $n = 1$, $4^n - 1$ is divisible by 3 for all $n \geqslant 1$.

1 for the case $n = 1$.
1 for the assumption.
1 for moving to $k + 1$.

Other strategies possible.

1 for a correct formulation.

1 for conclusion.

(The involvement of Σ not penalised.) Total 5

A8.

$$\frac{x^2}{(x+1)^2} = A + \frac{B}{x+1} + \frac{C}{(x+1)^2} \text{ so}$$

$$x^2 = A(x+1)^2 + B(x+1) + C$$
$$= Ax^2 + (2A + B)x + A + B + C$$

Hence $A = 1$, $B = -2$ and $C = 1$.

1 for valid method

2E1 for the values

(a) $$y = 1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}$$

so there is a vertical asymptote $x = -1$ and a horizontal asymptote $y = 1$.

1 for vertical asymptote
1 for horizontal asymptote

(b) $$\frac{dy}{dx} = \frac{2}{(x+1)^2} - \frac{2}{(x+1)^3} = 0 \text{ at SV}$$

$$\Rightarrow (x+1) = 1 \Rightarrow x = 0, y = 0$$

$$\frac{d^2y}{dx^2} = \frac{-4}{(x+1)^3} + \frac{6}{(x+1)^4}$$

$$= -4 + 6 \text{ when } x = 0$$

Thus $(0, 0)$ is a minimum.

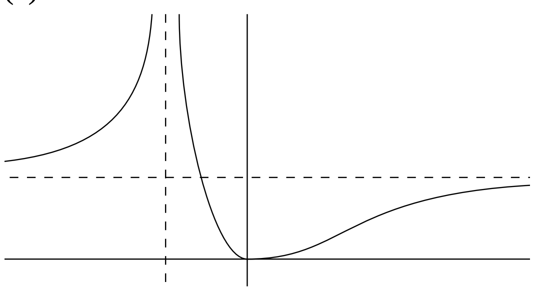
1 for derivative (however obtained)

1 for solving

1 for justification

1 for $(0, 0)$ is a minimum

(c)



1 for asymptotes
1 for branches

or 1 for each branch

Total 11

A9.	(a)	
	$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-xy^2}{-x^2y} = \frac{y}{x}$$	1 mark
	$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$	1 mark
	$$\ln y = \ln x + C$$	1 mark
	$$x = 1,\ y = 2 \Rightarrow C = \ln 2$$	1 mark for evaluating C
	$$\ln y = \ln x + \ln 2$$	
	$$y = 2x$$	1 mark for formula
	(b)	
	$$\frac{dx}{dt} = -x^2(2x) = -2x^3$$	1 mark
	$$\int \frac{1}{x^3}\,dx = \int -2\,dt$$	1 mark
	$$\frac{x^{-2}}{-2} = -2t + D$$	1 mark
	$$\frac{1}{x^2} = 4t - 2D$$	
	$$t = 0,\ x = 1 \Rightarrow D = -\frac{1}{2}$$	1 mark
	$$\frac{1}{x^2} = 4t + 1$$	
	$$x = \frac{1}{\sqrt{4t + 1}}$$	1 mark Total 10

A10.	$S_n(1) = 1 + 2 + 3 + \dots + n$ $= \tfrac{1}{2}n(n+1)$	1 for recognising that $S_n(1)$ requires special treatment. 1 for evaluating it correctly.
	$(1-x)S_n(x) = S_n(x) - xS_n(x)$ $= 1 + 2x + 3x^2 + \dots + nx^{n-1}$ $-\left(x + 2x^2 + 3x^3 + \dots + nx^n\right)$ $= 1 + x + x^2 + \dots + x^{n-1} - nx^n$	3E1 for expanding correctly and simplifying
	$= \dfrac{1-x^n}{1-x} - nx^n.$	1 for applying the sum of a GP
	Thus $S_n(x) = \dfrac{1-x^n}{(1-x)^2} - \dfrac{nx^n}{(1-x)}$ as required.	
	$\dfrac{2}{3} + \dfrac{3}{3^2} + \dfrac{4}{3^3} + \dots + \dfrac{n}{3^{n-1}} + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$ $= \left(S_n(\tfrac{1}{3}) - 1\right) + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$	1 for recognising that it relates to $S_n(\tfrac{1}{3})$.
	$= \dfrac{1-\frac{1}{3^n}}{(1-\frac{1}{3})^2} - \dfrac{n\frac{1}{3^n}}{1-\frac{1}{3}} - 1 + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$	1 for applying earlier result.
	$= \dfrac{9}{4}\left(1 - \dfrac{1}{3^n}\right) - \dfrac{3}{2}\cdot\dfrac{n}{3^n} - 1 + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$ $= \dfrac{5}{4}\left(1 - \dfrac{1}{3^n}\right)$	
	$\lim\limits_{n\to\infty}\left\{\dfrac{2}{3} + \dfrac{3}{3^2} + \dfrac{4}{3^3} + \dots + \dfrac{n}{3^{n-1}} + \dfrac{3}{2}\cdot\dfrac{n}{3^n}\right\}$ $= \dfrac{5}{4}$	1 for obtaining the limit. Total 9

SECTION B (Mathematics 3)

	Solution	Marks
B1.	(a) $\overrightarrow{AB} = 2\mathbf{i} - \mathbf{k}$; $\overrightarrow{AC} = \mathbf{i} - \mathbf{j} - 3\mathbf{k}$	1 for the two initial vectors
	$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 1 & -1 & -3 \end{vmatrix}$	1 for a cross product; Vector form acceptable.
	$= -\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$	1 for the normal vector
	Equation of π_1 is of the form $-x + 5y - 2z = c$ $(1,1,0) \Rightarrow c = -1 + 5 = 4$ So an equation is $-x + 5y - 2z = 4$	1 for the equation
	(b) Normals are $-\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.	1 for normals
	So the angle between the planes is given by $\cos^{-1}\left(\dfrac{-1 + 10 - 2}{\sqrt{30}\sqrt{6}}\right)$	1 for applying the scalar product
	$= \cos^{-1}\dfrac{7}{6\sqrt{5}}\ [\approx 58.6°]$	1 for result (must be acute) — Total 7
B2.	$A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}$. When $n = 1$, RHS $= \begin{pmatrix} 1+1 & 1 \\ -1 & 1-1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = A$. Therefore true when $n = 1$.	1 mark for showing true when $n = 1$
	Assume $A^k = \begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix}$.	1 for stating the assumption
	Consider A^{k+1}. $A^{k+1} = A.A^k$	1 for considering $k + 1$
	$= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix}$ $= \begin{pmatrix} k+2 & k+1 \\ -(k+1) & -k \end{pmatrix}$	1 for this matrix
	$= \begin{pmatrix} (k+1)+1 & (k+1) \\ -(k+1) & 1-(k+1) \end{pmatrix}$	1 for obtaining final matrix
	Thus if true for k then true for $k + 1$. Since true for $n = 1$, by induction, true for all $n \geqslant 1$.	1 for conclusion — Total 6

B3.	$f(x) = \ln(\cos x) \qquad f(0) = 0$ $f'(x) = \frac{-\sin x}{\cos x} = -\tan x \qquad f'(0) = 0$ $f''(x) = -\sec^2 x \qquad f''(0) = -1$ $f'''(x) = -2\sec^2 x \tan x \qquad f'''(0) = 0$ $f''''(x) = -4\sec^3 x \tan^2 x$ $\quad -2\sec^4 x \qquad f''''(0) = -2$ $f(x) = f(0) + xf'(0) + \ldots$ $\ln(\cos x) = 0 + 0.x - 1.\dfrac{x^2}{2} + 0.x - 2.\dfrac{x^4}{4!}$ $= -\dfrac{x^2}{2} - \dfrac{x^4}{12} + \ldots$	1 for first two derivatives 1 for third and fourth derivatives 1 for evaluation at 0 1 method mark for series Using series for log and cos can gain full marks. 1 for an expansion Total 5
B4.	$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $B = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} = \dfrac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$ $BA\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $= \dfrac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{2}\begin{pmatrix} \sqrt{3}x + y \\ x - \sqrt{3}y \end{pmatrix}$ i.e. $(x, y) \to \frac{1}{2}(\sqrt{3}x + y, x - \sqrt{3}y)$ so $k = \sqrt{3}$.	1 for A 1 for B 1 method for tackling a compostion 1 for value of k Total 4

B5.	$\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 4\cos x$	
	A.E. is $m^2 + 2m + 5 = 0$	1 for auxiliary equation
	$\Rightarrow m = -1 \pm 2i$	1 for roots
	C. F. is $y = e^{-x}\left(A\cos 2x + B\sin 2x\right)$	1 for form of complementary function
	For P.I. try $f(x) = a\cos x + b\sin x$	
	$f'(x) = -a\sin x + b\cos x$	
	$f''(x) = -a\cos x - b\sin x$	1 for derivatives
	Thus	
	$(4a+2b)\cos x + (4b-2a)\sin x = 4\cos x$	1 for substitution (Use of a wrong PI loses 2 of these 3 marks.)
	$\Rightarrow a = 2b \Rightarrow 10b = 4$	
	$\Rightarrow b = \frac{2}{5}$ and $a = \frac{4}{5}$	1 for values
	$y(x) = e^{-x}\left(A\cos 2x + B\sin 2x\right) + \frac{2}{5}\left(2\cos x + \sin x\right)$	
	$y(0) = 0 \Rightarrow A + \frac{4}{5} = 0 \Rightarrow A = -\frac{4}{5}$	1 for value of A
	$y'(x) = e^{-x}\left(-2A\sin 2x + 2B\cos 2x\right) - e^{-x}\left(A\cos 2x + B\sin 2x\right) + \frac{2}{5}\left(\cos x - 2\sin x\right)$	1 for derivative
	$y'(0) = 1 \Rightarrow 2B - A + \frac{2}{5} \Rightarrow B = -\frac{1}{10}$	1 for value of B
	$y = \dfrac{e^{-x}}{10}(-8\cos 2x - \sin 2x) + \dfrac{2}{5}(2\cos x + \sin x)$	1 for final statement
		Total 10

SECTION C (Statistics 1)

C1. Quota sampling, **1**

One advantage of this method is that no sampling frame is required. **1**
One disadvantage is that it can lead to biased results. **1**

C2. P (B born same day as A and C born same day as A) = 1/7 × 1/7 = 1/49. **1**

1/365 × 1/365 = 1/133225 **1**
= 133224 to 1 (which is well away from 160 000 to 1) so disagree **1**

50000 × 1/33225 =0.375 **1**
so about once every 3 years (or 3 times in 8 years) **1**

C3. Assume that standard deviation is still 28 seconds. **1**

$$\left.\begin{array}{l} H_0 : \mu = 453 \\ H_1 : \mu \neq 453 \end{array}\right\}$$ **1**

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{442 - 453}{28 / \sqrt{50}} = -2.78$$ **1**

$\mathrm{P}(z < -2.78) = \Phi(-2.78) = 1 - 0.9973 = 0.0027$
so that the p-value = $2 \times 0.0027 = 0.0054$ **1**

$0.0054 < 0.01$ so reject H_0 at the 1% level. **1**
i.e. there is evidence that the mean service time has changed. **1**

C4.

$$\hat{p} \pm 2.58\sqrt{\frac{\hat{p}\hat{q}}{n}}$$ **1**

$$= \frac{80}{250} \pm 2.58\sqrt{\frac{\frac{80}{250} \times \frac{170}{250}}{250}}$$ **1**

$$= 0.32 \pm 0.08 \left(\text{or } 0.24 \rightarrow 0.40\right)$$ **1**

In the long term 99 out of 100 of intervals calculated using 99% confidence will contain the ‘true’ value of the parameter being estimated. **1**

C5. The number of shock reactions in groups of 1200 will have the binomial distribution with parameters $n = 1200$ and $p = 1/2000$. **1**

This distribution can be approximated by the Poisson distribution with parameter $1200 \times 1/2000 = 0.6$. **1**

$\mathrm{P}(X > 2) = 1 - \mathrm{P}(X \leqslant 2)$ **1**

$= 1 - \left(e^{-0.6}.\dfrac{0.6^0}{0!} + e^{-0.6}.\dfrac{0.6^1}{1!} + e^{-0.6}.\dfrac{0.6^2}{2!}\right)$

$= 1 - (0.54881 + 0.32929 + 0.09879)$

$= 1 - 0.97689 = 0.023$ **1**

(arithmetical working not required as could be done on a Ti83)

C6. (a) $\mathrm{P}(X < 25)$ **1**

$= \mathrm{P}\left(Z < \dfrac{25 - 25.5}{0.4}\right)$

$= \mathrm{P}(Z < -1.25)$ **1**

$= 1 - \Phi(1.25) = 1 - 0.8944 = 0.1056$ **1**

(b) The distribution of the number of underweight bags is binomial **1**
with parameters $n = 40$ and $p = 0.1056$. **1**
P(No underweight bags) = ${}^{40}C_0\,(0.1056)^0\,(0.8944)^{40} = 0.0115$ **1**

(c) The mean weight of the sample of 40 bags must be less than 25 kg. **1**

$\mathrm{P}(\overline{X} < 25) = \mathrm{P}\left(Z < \dfrac{25 - 25.5}{0.4/\sqrt{40}}\right)$ **1**

$= \mathrm{P}(Z < -7.9)$ **1**

≈ 0 **1**

or

$T \sim N(1020,\ 6.4)$

and $P(T < 1000) = P(Z < -7.9)$

$\approx 0.$

SECTION D (Numerical Analysis 1)

D1. $$L(x) = \frac{(x-1.3)(x-1.8)}{(-0.3)(-0.8)}0.758 + \frac{(x-1.0)(x-1.8)}{(0.3)(-0.5)}1.106 + \frac{(x-1.0)(x-1.3)}{(0.8)(0.5)}0.994$$

$$= \left(x^2 - 3.1x + 2.34\right)3.158 - \left(x^2 - 2.8x + 1.8\right)7.373 + \left(x^2 - 2.3x + 1.3\right)2.485$$

$$= -1.730x^2 + 5.139x - 2.651$$ **4**

D2. $f(x) = \sin 2x$; $f'(x) = 2\cos 2x$; $f''(x) = -4\sin 2x$;

$f'''(x) = -8\cos 2x$; $f^{\text{iv}}(x) = 16\sin 2x$

Taylor polynomial is

$$p\left(\frac{\pi}{4} + h\right) = \sin\frac{\pi}{2} - \frac{4h^2}{2}\sin\frac{\pi}{2} + \frac{16h^4}{24}\sin\frac{\pi}{2}$$

$$= 1 - 2h^2 + \tfrac{2}{3}h^4$$ **3**

$\sin 96° = \sin(\pi/2 + \pi/30)$ and $h = \pi/60$

Second degree approximation is

$$1 - 2\left(\frac{\pi}{60}\right)^2 = 1 - 0.0055 = 0.9945$$ **2**

Principal truncation error term is $\tfrac{2}{3}\left(\tfrac{\pi}{60}\right)^4 \approx 0.000005$

Hence second order estimate should be accurate to 4 decimal places. **2**

D3. Let quadratic through $(x_0, f_0), (x_1, f_1), (x_2, f_2)$ be

$$y = A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1).$$

Then $f_0 = A_0$; $f_1 = A_0 + A_1 h$; $f_2 = A_0 + 2A_1 h + 2A_2 h^2$

and so $A_1 = \dfrac{f_1 - f_0}{h} = \dfrac{\Delta f_0}{h}$; $A_2 = \dfrac{f_2 - 2f_1 + f_0}{2h^2} = \dfrac{\Delta^2 f_0}{2h^2}$.

Thus $y = f_0 + \dfrac{x - x_0}{h}\Delta f_0 + \dfrac{(x - x_0)(x - x_1)}{2h^2}\Delta^2 f_0$.

Setting $x = x_0 + ph$, where $0 < p < 1$, gives

$$y = f_0 + p\Delta f_0 + \tfrac{1}{2}p(p-1)\Delta^2 f_0$$ **5**

(Can also be done by an operator expansion of $(1 + \Delta)^p$.)

D4. (a) Maximum error is 8ε. i.e. $8 \times 0.0005 = 0.004$. **1**

(b) $\Delta^2 f_1 = 0.045$ **1**

(c) Third degree polynomial would be suitable (constant differences). **1**

(d) Working from $x = 3.2$, $p = 0.1$

$$f(0.321) = 0.459 + 0.1(0.224) + \frac{(0.1)(-0.9)}{2}(0.051) + \frac{(0.1)(-0.9)(-1.9)}{6}(0.009)$$

$$= 0.459 + 0.022 - 0.002 + 0.000 = 0.479$$ **3**

D5. (a) Simpson's rule calculation is:

x	$f(x)$	m_1	$m_1 f_1(x)$	m_2	$m_2 f_2(x)$
1	0.4657	1	0.4657	1	0.4657
1.5	0.8320			4	3.3280
2	1.1261	4	4.5044	2	2.2522
2.5	1.0984			4	4.3936
3	0.8238	1	0.8238	1	0.8238
			5.7939		11.2633

Hence $I_2 = 5.7939/3 = 1.9313$

and $I_4 = 11.2633 \times 0.5/3 = 1.8772$

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(b) Maximum truncation error $\approx 2 \times 0.324/180 = 0.0036$ **1**

Hence suitable estimate is $I_4 = 1.88$ **1**

(c) With n strips and step size $2h$, the Taylor series for expansion of an integral approximated by Simpson's rule (with principal truncation error of $O(h^2)$) is

$$I = I_n + C(2h)^4 + D(2h)^6 + \ldots = I_n + 16Ch^4 \qquad (1)$$

With $2n$ strips and step size h, we have

$$I = I_{2n} + Ch^4 + Dh^6 + \ldots \qquad (2)$$

$16 \times (2) - (1)$ gives $15I = 16I_{2n} - I_n + O(h^6)$

i.e. $I \approx (16I_{2n} - I_n)/15 = I_{2n} + (I_{2n} - I_n)/15$ **3**

$I_3 = 1.8772 + (1.8772 - 1.9313)/15 = 1.8736$
or 1.874 to suitable accuracy **1**

SECTION E (Mechanics 1)

E1. (a) From the equation of motion for the vertical motion

$$\dot{y} = V \sin 45^\circ - gt = \frac{1}{\sqrt{2}}V - gt. \qquad \mathbf{1}$$

The shell attains its maximum height when

$$\dot{y} = 0 \Rightarrow V = \sqrt{2}\, gt = 69.3 \text{ m s}^{-1}. \qquad \mathbf{1}$$

(b) The shell hits the ground again after 10 seconds. From the equation of motion for horizontal motion

$$x = V \cos 45^\circ t = \frac{1}{\sqrt{2}} Vt. \qquad \mathbf{1}$$

The range is

$$R = \frac{1}{\sqrt{2}} Vt \approx 490 \text{ m}. \qquad \mathbf{1}$$

E2. (a) The position of the car is

$$x_C = \tfrac{1}{2} at^2, \qquad \mathbf{1}$$

and the position of the lorry

$$x_L = Ut + \tfrac{1}{4} at^2. \qquad \mathbf{1}$$

When the car and the lorry draw level

$$x_C = x_L \qquad \mathbf{1}$$

$$\Leftrightarrow t\left(\tfrac{1}{4}at - U\right) = 0$$

$$\Leftrightarrow t = 0 \text{ or } t = \frac{4U}{a}$$

and as $t > 0$ we take $t = \dfrac{4U}{a}$. **1**

(b) When the car draws level with the lorry it has travelled

$$x_C = \frac{1}{2} a \left(\frac{4U}{a}\right)^2 = \frac{8U^2}{a}. \qquad \mathbf{1}$$

E3. (a) Resolving perpendicular to the plane

$$N + P \sin 30^\circ = mg \cos 30^\circ \qquad \mathbf{1}$$

$$\Rightarrow N = \sqrt{3}\, g - \tfrac{1}{2}P$$

$$= \tfrac{1}{2}\left(2\sqrt{3}\, g - P\right). \qquad \mathbf{1}$$

The frictional force is

$$F = \mu N = \tfrac{1}{4}\left(2\sqrt{3}\, g - P\right). \qquad \mathbf{1}$$

(b) Resolving parallel to the plane and using Newton II

$$P \cos 30^\circ = mg \sin 30^\circ + F \qquad \mathbf{1}$$

$$\Leftrightarrow \frac{\sqrt{3}}{2}P = g + \frac{1}{4}\left(2\sqrt{3}\, g - P\right)$$

$$\Leftrightarrow \tfrac{1}{2}\left(\sqrt{3} + \tfrac{1}{2}\right)P = \left(1 + \tfrac{1}{2}\sqrt{3}\right)g \qquad \mathbf{1}$$

$$\Leftrightarrow P = \frac{2\left(2 + \sqrt{3}\right)g}{\left(2\sqrt{3} + 1\right)} \approx 16{\cdot}4 \text{ N}. \qquad \mathbf{1}$$

E4. (a) Resolving forces horizontally gives

$$T_1 \cos 30° = T_2 \cos 60° \quad \mathbf{1}$$

$$\Rightarrow \frac{\sqrt{3}}{2} T_1 = \frac{1}{2} T_2$$

$$\Rightarrow T_2 = \sqrt{3}\, T_1 > T_1. \quad \mathbf{1}$$

(b) Resolving forces vertically and using Newton II

$$ma = T_1 \sin 30° + T_2 \sin 60° - mg \quad \mathbf{1}$$

$$\Rightarrow \frac{1}{2} T_1 + \frac{\sqrt{3}}{2} T_2 = m(a + g) \quad \mathbf{1}$$

$$\frac{1}{2} \frac{1}{\sqrt{3}} T_2 + \frac{\sqrt{3}}{2} T_2 = m(a + g) \quad \mathbf{1}$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{3}} + \sqrt{3}\right) T_2 = m(a + g)$$

$$\Rightarrow T_2 = \frac{\sqrt{3}}{2} m(a + g) \quad \mathbf{1}$$

E5. (a) Since $\mathbf{a}_A = -\frac{2}{5}t\mathbf{i}$, $\mathbf{v}_A(t) = -\frac{1}{5}t^2\mathbf{i} + \mathbf{c}$.

Since $\mathbf{v}_A(0) = 10\mathbf{i}$, we have $\mathbf{c} = 10\mathbf{i}$ so

$$\mathbf{v}_A(t) = \left(10 - \tfrac{1}{5}t^2\right)\mathbf{i} \quad \mathbf{1}$$

Integrating again gives

$$\mathbf{r}_A(t) = \left(10t - \tfrac{1}{15}t^3\right)\mathbf{i} + \mathbf{c}_2$$

but since $\mathbf{r}(0) = \mathbf{0}$ then $\mathbf{c}_2 = \mathbf{0}$ and

$$\mathbf{r}_A(t) = \frac{t}{15}\left(150 - t^2\right)\mathbf{i} \quad \mathbf{1}$$

(b)(i)

$$\dot{\mathbf{r}}_B = \tfrac{1}{15}\left\{75 - 3t^2\right\}\mathbf{i} = \mathbf{0} \text{ when} \quad \mathbf{1}$$

$$3t^2 = 75. \quad \mathbf{1}$$

$$t = 5$$

When $t = 5$ $\quad \mathbf{r}_B = \tfrac{1}{15}\left\{45 + 375 - 125\right\}\mathbf{i} + 4\mathbf{j}$

$$= \tfrac{59}{3}\mathbf{i} + 4\mathbf{j}. \quad \mathbf{1}$$

So the distance from the origin = $\sqrt{\left(\frac{59}{3}\right)^2 + 4^2} \approx 20.1$ m $\quad \mathbf{1}$

(ii)

$$\mathbf{r}_A - \mathbf{r}_B = \tfrac{1}{15}t\left(150 - t^2\right)\mathbf{i} - \tfrac{1}{15}\left(45 + 75t - t^3\right)\mathbf{j} - 4\mathbf{j}$$

$$= \tfrac{1}{15}\left(75t - 45\right)\mathbf{i} - 4\mathbf{j} = \left(5t - 3\right)\mathbf{i} - 4\mathbf{j} \quad \mathbf{1}$$

$$|\mathbf{r}_A - \mathbf{r}_B|^2 = (5t - 3)^2 + 16 \quad \mathbf{1}$$

To find the minimum value

$$\frac{d}{dt}\left(|\mathbf{r}_A - \mathbf{r}_B|^2\right) = 2(5t - 3) \times 5 = 0 \quad \mathbf{1}$$

so the minimum occurs when $t = \frac{3}{5}$. $\quad \mathbf{1}$

The minimum distance is then $\sqrt{16} = 4$ m. $\quad \mathbf{1}$

[END OF MARKING INSTRUCTIONS]