Section B (Numerical Analysis 1 and 2)

Marks

ONLY candidates doing the course Numerical Analysis 1 and 2 and one unit chosen from Mathematics 1 (Section D), Statistics 1 (Section E) and Mechanics 1 (Section G) should attempt this Section.

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

B1. The function f is defined for $x > 0 \cdot 8$ by $f(x) = \dfrac{1}{5x-4}$.

The polynomial p is the Taylor polynomial of degree two for the function f near $x = 1$. Express $p(1+h)$ in the form $c_0 + c_1 h + c_2 h^2$. **3**

Use this polynomial to estimate the value of $f(0{\cdot}99)$. **2**

State, with a reason, whether or not $f(x)$ is sensitive to small changes in x in the neighbourhood of $x = 1$. **1**

B2. The following data are available for a function f:

x	0	2	5
$f(x)$	1·3271	1·5238	1·8516

Use the quadratic Lagrange interpolation formula to estimate $f(3)$. **3**

B3. In the usual notation for forward differences of function values $f(x)$ tabulated at equally spaced values of x,

$$\Delta f_i = f_{i+1} - f_i,$$

where $f_i = f(x_i)$ and $i = \ldots -2, -1, 0, 1, 2, \ldots$

Show that $\Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0$. **2**

If each value of f_i is subject to an error whose magnitude is less than or equal to ε, determine the magnitude of the maximum possible rounding error in $\Delta^3 f_0$. **1**

Rounded values of a function f are known to be $f_0 = 1{\cdot}311$, $f_1 = 1{\cdot}416$, $f_2 = 1{\cdot}532$, $f_3 = 1{\cdot}658$. Obtain $\Delta^3 f_0$ and the magnitude of the maximum rounding error in $\Delta^3 f_0$. **2**

Hence state whether or not this third difference appears to be significantly different from zero. **1**

B4. The following data (accurate to the degree implied) are available for a function f:

x	1·0	1·1	1·2	1·3	1·4	1·5
$f(x)$	1·263	1·456	1·696	1·991	2·351	2·782

(*a*) Construct a difference table of fourth order for the data. **3**

(*b*) Using the Newton forward difference formula of degree three, and working to three decimal places, obtain an approximation to $f(1{\cdot}18)$. **3**

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B5. The sequence $\{a_r\}$ is defined by the recurrence relation

$$5a_{n+1} + 3a_n = 1, \qquad \text{with } a_0 = 2.$$

Trace the sequence as far as the term a_4. (Work to two decimal places where appropriate.) **2**

Display the convergence of the sequence using a cobweb diagram, ie a sketch based on drawing the line segments between the points with coordinates (a_0, a_0), (a_0, a_1), (a_1, a_1), (a_1, a_2), (a_2, a_2), etc. **3**

Determine the x-coordinate of the point at the centre of the cobweb diagram, ie the fixed point of the sequence. **1**

B6. Express the polynomial $f(x) = 3{\cdot}1x^4 + 2{\cdot}8x^3 - 4{\cdot}2x + 2{\cdot}0$ in nested form and evaluate $f(1{\cdot}2)$ to two decimal places. **2**

Given that the coefficients of $f(x)$ are rounded to the accuracy implied, determine the maximum possible value of $f(1{\cdot}2)$. **1**

B7. Write the following equations in diagonally dominant form:

$$\begin{aligned} 0{\cdot}71x_1 + 2{\cdot}78x_2 - 0{\cdot}08x_3 &= 3{\cdot}18 \\ 0{\cdot}46x_1 - 0{\cdot}34x_2 + 5{\cdot}17x_3 &= 9{\cdot}25 \\ 4{\cdot}24x_1 + 0{\cdot}27x_2 - 0{\cdot}46x_3 &= -1{\cdot}14. \end{aligned}$$ **1**

Explain why this is necessary when the equations are to be solved using an iterative procedure. **1**

Use the Jacobi iterative procedure, with $x_1 = x_2 = x_3 = 0$ as a first approximation, to obtain the *first iterates* of x_1 and x_2 for the solution of these equations. **1**

B8. (*a*) Using a Taylor polynomial of degree two, or otherwise, derive the trapezium rule over a single strip and the corresponding principal error term. **5**

(*b*) Use the composite trapezium rule with four strips to obtain an estimate for the integral

$$\int_1^{1{\cdot}4} x^2 \ln x \, dx.$$

Perform the calculations using four decimal places. **2**

(*c*) Given that for $f(x) = x^2 \ln x$, $f''(x) = 2\ln x + 3$, obtain the maximum value of $x^2 \ln x$ on the interval $[1, 1{\cdot}4]$ and hence obtain an estimate of the maximum truncation error in the integral. **3**

Hence state the value of the integral to a suitable accuracy. **1**

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B9. The system of linear equations

$$\begin{pmatrix} -2\cdot 8 & 1\cdot 5 & 0\cdot 7 \\ 3\cdot 7 & -4\cdot 2 & 2\cdot 2 \\ 0\cdot 0 & 2\cdot 6 & 7\cdot 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 \\ 8\cdot 4 \\ -2\cdot 8 \end{pmatrix}$$

has arisen in the solution of a problem.

Use Gaussian elimination with partial pivoting to solve the system, incorporating a row check and carrying two guard figures in the calculation. Record your answers rounded to one decimal place. **6**

B10. The equation $f(x) = 0$ has a root close to $x = x_0$. By drawing a suitable graph to illustrate this situation, derive the formula for the first iteration of the Newton-Raphson method of solution of $f(x) = 0$. Hence explain how the general formula is obtained. **3**

It is known that the equation $f(x) = 0$, where $f(x) = e^x - 3x^2 - 1$, has three distinct real roots one of which lies in the interval [3, 4].

Given that $\frac{d}{dx}(e^x) = e^x$, use the Newton-Raphson method to determine this root correct to three decimal places. **3**

Explain the term *ill-conditioning* in relation to the determination of the roots of the equation $f(x) = 0$. It is known that the other two roots lie in the narrow interval [0, 0·5].

Use a diagram to explain why the Newton-Raphson method may be difficult to use in the determination of these roots. **3**

B11. A predictor-corrector method for the solution of a differential equation uses Euler's method as predictor. Explain, with the aid of a diagram, how the trapezium rule may be used as the corrector. **3**

Use this predictor-corrector method with two applications of the corrector to obtain a solution at $x = 1\cdot 1$ of the differential equation

$$\frac{dy}{dx} = (x^2 - y + 1)\cos x, \quad y(1) = 1.$$

Use step size 0·1 and carry four decimal places in the calculation. **6**

[*END OF SECTION B*]

All candidates who have attempted Section B (Numerical Analysis 1 and 2) should now attempt ONE of the following

Section D (Mathematics 1) on *Page fourteen*

Section E (Statistics 1) on *Pages fifteen* and *sixteen*

Section G (Mechanics 1) on *Pages nineteen* and *twenty.*