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Matrices and Systems of Equations - Lesson 5

Determinants and Inverses of 2 x 2 Matrices

LI

- Calculate 2 x 2 Determinants.
- Calculate the Inverse of a 2 x 2 matrix.
- Solve systems of equations using inverses.

SC

- Primary school arithmetic.

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Solving the simultaneous equations,

$$a x + b y = e$$
$$c x + d y = f$$

gives,

$$x = \frac{d e - b f}{a d - b c} \quad , \quad y = \frac{a f - c e}{a d - b c}$$

For solutions to exist, we must therefore have $a d - b c \neq 0$; if not, then no solutions.

The above system of equations can be written as,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

The determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is,

$$\det A = |A| = a d - b c$$

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A matrix A has an inverse (or A is invertible or A is non-singular) if there is a matrix B $(= A^{-1})$ satisfying,

$$AB = I_n = BA$$

$$(AA^{-1} = I_n = A^{-1}A)$$

A, B both square matrices of order $n \times n$

A is an $n \times n$ matrix	
A is invertible	A is non-invertible
$\Updownarrow$	$\Updownarrow$
$\det A \neq 0$	$\det A = 0$
$\Updownarrow$	$\Updownarrow$
A is non-singular	A is singular

For a 2 x 2 matrix, the previous table becomes :

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A is invertible	A is non-invertible
$\Updownarrow$	$\Updownarrow$
$ad - bc \neq 0$	$ad - bc = 0$
$\Updownarrow$	$\Updownarrow$
A is non-singular	A is singular
$\Updownarrow$	$\Updownarrow$
$ax + by = e$ $cx + dy = f$ has solutions	$ax + by = e$ $cx + dy = f$ has no solutions

The inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is,

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example 1

Determine whether or not $A = \begin{pmatrix} 3 & -3 \\ -2 & -1 \end{pmatrix}$ is invertible.

$$|A| = (3 \times (-1)) - ((-3) \times (-2))$$

$$\Rightarrow \underline{|A| = -9}$$

$$\boxed{\text{As } |A| \neq 0, A \text{ is invertible}}$$

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Example 2

Find the value(s) of k for which the following system of equations has no solutions.

$$4x - 2ky = -1$$
$$kx - 3y = 0$$

Let $B = \begin{pmatrix} 4 & -2k \\ k & -3 \end{pmatrix}$. For no solutions, $\det B = 0$.

$$\det B = 0$$
$$\therefore \quad (4 \times (-3)) - ((-2k) \times k) = 0$$
$$\Rightarrow \quad -12 + 2k^2 = 0$$
$$\Rightarrow \quad k^2 = 6$$
$$\Rightarrow \quad \boxed{k = \pm\sqrt{6}}$$

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Example 3

Find Q^{-1} if $Q = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$.

$$|Q| = (2 \times 3) - (4 \times 1)$$

$$\Rightarrow \quad \underline{|Q| = 2}$$

$$\therefore \quad \boxed{Q^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}}$$

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Example 4

For what value of k is the matrix,

$$\begin{pmatrix} 1 & 2 \\ 6 - k & k \end{pmatrix}$$

singular ?

Denoting the matrix by W, we require,

$$|W| = 0$$

$$\therefore \quad (1 \times k) - (2 \times (6 - k)) = 0$$

$$\Rightarrow \quad k - 12 + 2k = 0$$

$$\Rightarrow \quad 3k = 12$$

$$\Rightarrow \quad \boxed{k = 4}$$

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Example 5

A non-singular matrix F satisfies $F^2 = 3F - 2I$, where I is the identity matrix.

Show that $F^{-1} = \frac{3}{2}I - \frac{1}{2}F$.

$$F^2 = 3F - 2I$$

$$\therefore \quad F^{-1}F^2 = F^{-1}(3F - 2I)$$

$$\Rightarrow \quad F = 3F^{-1}F - 2F^{-1}I$$

$$\Rightarrow \quad F = 3I - 2F^{-1}$$

$$\Rightarrow \quad 2F^{-1} = 3I - F$$

$$\Rightarrow \quad \boxed{F^{-1} = \frac{3}{2}I - \frac{1}{2}F}$$

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Example 6

Use matrix multiplication to solve,

$$\begin{aligned} 2x + 3y &= 1 \\ x - y &= 3 \end{aligned}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \star$$

Letting $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$, we have $|A| = -5$ and hence,

$$A^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix}$$

Pre-multiplying $\star$ by A^{-1} gives,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -10 \\ 5 \end{pmatrix}$$

$$\Rightarrow \quad \underline{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}}$$

$$\boxed{x = 2, y = -1}$$

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AH Maths - MiA (2nd Edn.)

- pg. 240-1 Ex. 13.6 All Q.
- pg. 243-4 Ex. 13.7
  Q 1, 3 a - e, 5, 6 a, 7 - 10, 11 a.

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Ex. 13.6

1 Calculate the determinant of each matrix.

a $\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$ b $\begin{pmatrix} 12 & 3 \\ 8 & 2 \end{pmatrix}$ c $\begin{pmatrix} 5 & -2 \\ -3 & -1 \end{pmatrix}$ d $\begin{pmatrix} 1 & 0 \\ 3 & 7 \end{pmatrix}$

e $\begin{pmatrix} -4 & -5 \\ 2 & -1 \end{pmatrix}$ f $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ g $\begin{pmatrix} 7 & 1 \\ -2 & -1 \end{pmatrix}$ h $\begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$

2 Find x in each case.

a $\begin{vmatrix} 2 & 1 \\ 1 & x \end{vmatrix} = 7$ b $\begin{vmatrix} 3 & -1 \\ 2x & 4 \end{vmatrix} = 26$ c $\begin{vmatrix} 3x & 1 \\ 1 & x \end{vmatrix} = 47$ d $\begin{vmatrix} 2x & x \\ -3 & x \end{vmatrix} = 2$

e $\begin{vmatrix} x & x \\ 3 & x \end{vmatrix} = 10$ f $\begin{vmatrix} \sin x^\circ & -1 \\ 1 & 1 \end{vmatrix} = 0, 0 \le x \le 360$ g $\begin{vmatrix} e^x & e \\ 1 & e^x \end{vmatrix} = 0$

3 For each of these systems of equations

i express the system of equations as a matrix equation of the form $AX = B$ where $X = \begin{pmatrix} x \\ y \end{pmatrix}$

ii by considering $\det(A)$, determine whether the system has a solution.

a $x + y = 1$, $3x + 2y = 4$ b $4x + y = 2$, $8x + 2y = 3$ c $5x - 2y = 2$, $10x - 4y = 1$ d $5x - y = 1$, $3x - 2y = 0$

4 Study each system of equations and say for what value(s) of k the system has no solution.

a $3x + ky = 3$, $x + 2y = 1$ b $5x - 2ky = 4$, $3x + 6y = -2$ c $6x - 2ky = -1$, $kx - 12y = 0$ d $kx + (k - 4)y = 5$, $2x + ky = 3$

Ex. 13.7

1 Find A^{-1}, the inverse of the matrix $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, and verify that $A^{-1}A = I = AA^{-1}$.

3 Find the inverse, where it exists, of each of these matrices.

a $\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$ b $\begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}$ c $\begin{pmatrix} 9 & 6 \\ 3 & 2 \end{pmatrix}$ d $\begin{pmatrix} 4 & -2 \\ -7 & 3 \end{pmatrix}$ e $\begin{pmatrix} 9 & 2 \\ 13 & 3 \end{pmatrix}$

5 Given that $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix}$, show that

a $A^{-1}B^{-1} = (BA)^{-1}$ b $(AB)^{-1} = B^{-1}A^{-1}$ c $|A|\,|B| = |AB|$

6 a Given that A and B are square matrices, simplify

i $A^{-1}B^{-1}BA$ ii $BAA^{-1}B^{-1}$ iii $B^{-1}A^{-1}AB$ iv $ABB^{-1}A^{-1}$

7 Find the inverse of

a $\begin{pmatrix} a & -a \\ -a & 2a \end{pmatrix}$ b $\begin{pmatrix} 3x & 5x \\ 2x & 4x \end{pmatrix}$ c $\begin{pmatrix} 3t & t^3 \\ 2 & t^2 \end{pmatrix}$

d $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ e $\begin{pmatrix} 1-x & x \\ -x & 1+x \end{pmatrix}$ f $\begin{pmatrix} a & a+1 \\ a-1 & a \end{pmatrix}$

8 For what values of k are these matrices singular?

a $\begin{pmatrix} 2 & 2 \\ 4 & k \end{pmatrix}$ b $\begin{pmatrix} 1 & 2k \\ 3 & 6 \end{pmatrix}$ c $\begin{pmatrix} 1-k & -1 \\ 3 & 1+k \end{pmatrix}$ d $\begin{pmatrix} 2+k & -6 \\ 4 & 3-k \end{pmatrix}$

9 Show that these matrices are their own inverse (that is, $A^{-1} = A$).

a $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ b $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ c $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ d $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ e $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

f $\begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}$ g $\begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{pmatrix}$

10 $A = \begin{pmatrix} 2 & 3 \\ 7 & 11 \end{pmatrix}$

a Show that $A^2 = 13A - I$.

b Hence show (without evaluating A^3 or A^{-1}) that

i $A^3 = 168A - 13I$ ii $A^{-1} = 13I - A$

11 For each of these systems of equations

i express the system of equations in the form $AX = B$

ii find the inverse of the matrix A

iii hence solve the system for x and y.

a $2x + y = 9$
$3x + 2y = 16$

Answers to AH Maths (MiA), pg. 240-1, Ex. 13.6

1 **a** 14 **b** 0 **c** -11 **d** 7
e 14 **f** -1 **g** -5 **h** 1

2 **a** 4 **b** 7 **c** ± 4 **d** $-2, 0{\cdot}5$
e $-2, 5$ **f** 270 **g** $\frac{1}{2}$

3 **a** **i** $\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **ii** $|A| = -1 \Rightarrow \exists$ a solution

b **i** $\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **ii** No solution

c **i** $\begin{pmatrix} 5 & -2 \\ 10 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **ii** No solution

d **i** $\begin{pmatrix} 5 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **ii** $|A| = -7 \Rightarrow \exists$ a solution

4 **a** 6 **b** -5 **c** ± 6 **d** No k

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Answers to AH Maths (MiA), pg. 243-4, Ex. 13.7

1 $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

3 **a** No inverse **b** $\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$ **c** No inverse

d $-\frac{1}{2}\begin{pmatrix} 3 & 2 \\ 7 & 4 \end{pmatrix}$ **e** $\begin{pmatrix} 3 & -2 \\ -13 & 9 \end{pmatrix}$

5 **a** Proof **b** Proof **c** Proof

6 **a** **i** **ii** **iii** **iv** I

7 **a** $\frac{1}{a}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ **b** $\frac{1}{2x}\begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix}$

c $\frac{1}{t^3}\begin{pmatrix} t^2 & -t^3 \\ -2 & 3t \end{pmatrix}$ **d** $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

e $\begin{pmatrix} 1+x & -x \\ x & 1-x \end{pmatrix}$ **f** $\begin{pmatrix} a & -1-a \\ 1-a & a \end{pmatrix}$

8 **a** 4 **b** 1 **c** ± 2 **d** $-5, 6$

9 **a–g** Proofs

10 **a** Proof **b** **i** **ii** Proofs

11 **a** **i** $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$ **ii** $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

iii $(x, y) = (2, 5)$