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Unit 1 : Integral Calculus - Lesson 4

Integration by Parts

LI

- Integrate a product of functions.

SC

- Integration by Parts formula.

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A product of functions is integrated using Integration by Parts

The Product Rule for differentiating functions u and v (where the dash means differentiate wrt x) is,

$$(u\,v)' = u'\,v + u\,v'$$

Integrating both sides wrt x gives,

$$\int (u\,v)'\,dx = \int u'\,v\,dx + \int u\,v'\,dx$$

$$\therefore \quad u\,v = \int u'\,v\,dx + \int u\,v'\,dx$$

$$\Rightarrow \quad \boxed{\int u\,v'\,dx = u\,v - \int u'\,v\,dx}$$

(Integration by Parts formula)

For definite integrals,

$$\boxed{\int_a^b u\,v'\,dx = [u\,v]_a^b - \int_a^b u'\,v\,dx}$$

Thus, to integrate a product of functions, one of the functions must be chosen to be integrated, the other chosen to be differentiated

The choice is made in such a way that, generally speaking, the integral on the RHS of the above formulae is no more difficult to evaluate than the original integral

Some guiding principles ($n \in \mathbb{N}$, $m \in \mathbb{R}$) :

Integrand	Differentiate
$x^n \sin mx$ or $x^n \cos mx$	x^n
$x^n e^{mx}$	x^n
$x^n \ln(mx)$	$\ln(mx)$

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Example 1

Integrate $x \cos x$.

Let $I = \int \overset{u}{x} \overset{v'}{\cos x} \, dx.$

$$\frac{d}{dx} \Big\downarrow \quad \boxed{\begin{array}{lll} u = x, & v = \sin x \\ u' = 1, & v' = \cos x \end{array}} \quad \Big\uparrow \int$$

$$I = uv - \int u'v \, dx$$

$$\therefore \quad I = x \sin x - \int 1 . \sin x \, dx$$

$$\Rightarrow \quad I = x \sin x - \int \sin x \, dx$$

$$\Rightarrow \quad \boxed{I = x \sin x + \cos x + C}$$

Example 2

Integrate $x^2 \ln x$.

Let $I = \int \overset{v'}{x^2} \overset{u}{\ln x} \, dx.$

$$\frac{d}{dx} \Big\downarrow \quad \boxed{\begin{array}{lll} u = \ln x \ , & v = x^3/3 \\ u' = 1/x \ , & v' = x^2 \end{array}} \quad \Big\uparrow \int$$

$$I = uv - \int u' v \, dx$$

$$\therefore \quad I = (1/3)\, x^3 \ln x - (1/3) \int (1/x) \,.\, x^3 \, dx$$

$$\Rightarrow \quad I = (1/3)\, x^3 \ln x - (1/3) \int x^2 \, dx$$

$$\Rightarrow \quad \boxed{I = (1/3)\, x^3 \ln x - (1/9)\, x^3 + C}$$

Example 3

Integrate $x e^{2x}$.

Let $I = \int \overset{u}{x} \, \overset{v'}{e^{2x}} \, dx.$

$$\frac{d}{dx} \Big\downarrow \quad \boxed{\begin{array}{l} u = x \;, \quad v = e^{2x}/2 \\ u' = 1 \;, \quad v' = e^{2x} \end{array}} \quad \Big\uparrow \int$$

$$I = u v - \int u' v \, dx$$

$$\therefore \quad I = (1/2) x e^{2x} - (1/2) \int 1 . e^{2x} \, dx$$

$$\Rightarrow \quad I = (1/2) x e^{2x} - (1/2) \int e^{2x} \, dx$$

$$\Rightarrow \quad \boxed{I = (1/2) x e^{2x} - (1/4) e^{2x} + C}$$

Example 4 (Repeated Integration by Parts)

Integrate $x^2 \cos x$.

Let $I = \int \overset{u}{x^2} \overset{v'}{\cos x} \, dx.$

$$\frac{d}{dx} \Big\downarrow \quad \boxed{\begin{array}{lll} u = x^2 & , & v = \sin x \\ u' = 2x & , & v' = \cos x \end{array}} \quad \Big\uparrow \int$$

$$I = uv - \int u' v \, dx$$

$$\therefore \quad I = x^2 \sin x - 2 \int x \sin x \, dx$$

Let $J = \int \overset{U}{x} \overset{V'}{\sin x} \, dx.$

$$\frac{d}{dx} \Big\downarrow \quad \boxed{\begin{array}{lll} U = x & , & V = -\cos x \\ U' = 1 & , & V' = \sin x \end{array}} \quad \Big\uparrow \int$$

$$J = UV - \int U' V \, dx$$

$$\therefore \quad J = -x \cos x + \int \cos x \, dx$$

$$\Rightarrow \quad \underline{J = -x \cos x + \sin x}$$

$$I = x^2 \sin x - 2J$$

$$\therefore \quad I = x^2 \sin x - 2(-x \cos x + \sin x) + C$$

$$\Rightarrow \quad \boxed{I = x^2 \sin x + 2x \cos x - 2 \sin x + C}$$

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Example 5 (Cyclic Integral)

Integrate $e^{-x}\cos x$.

Let $I = \int \overset{v'}{e^{-x}} \overset{u}{\cos x}\, dx.$

$$\frac{d}{dx}\Big\downarrow \quad \boxed{\begin{array}{lll} u = \cos x & , & v = -e^{-x} \\ u' = -\sin x & , & v' = e^{-x} \end{array}} \quad \Big\uparrow \int$$

$$I = uv - \int u'v\, dx$$

$$\therefore \quad I = -e^{-x}\cos x - \int e^{-x}\sin x\, dx$$

Let $J = \int \overset{V'}{e^{-x}} \overset{U}{\sin x}\, dx.$

$$\frac{d}{dx}\Big\downarrow \quad \boxed{\begin{array}{lll} U = \sin x & , & V = -e^{-x} \\ U' = \cos x & , & V' = e^{-x} \end{array}} \quad \Big\uparrow \int$$

$$J = UV - \int U'V\, dx$$

$$\therefore \quad J = -e^{-x}\sin x + \int e^{-x}\cos x\, dx$$

$$\Rightarrow \quad \underline{J = -e^{-x}\sin x + I}$$

$$I = -e^{-x}\cos x - J$$

$$\therefore \quad I = -e^{-x}\cos x - (-e^{-x}\sin x + I)$$

$$\Rightarrow \quad I = -e^{-x}\cos x + e^{-x}\sin x - I$$

$$\Rightarrow \quad 2I = -e^{-x}\cos x + e^{-x}\sin x$$

$$\Rightarrow \quad \boxed{I = (1/2)\, e^{-x}(\sin x - \cos x) + C}$$

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Example 6 (The Hidden Function)

Find the exact value of $\int_0^1 \sin^{-1} x \; dx$.

Always integrate the 1

$$\text{Let } I = \int_0^1 \sin^{-1} x \; dx = \int_0^1 \overset{v'}{1} \,.\, \overset{u}{\sin^{-1} x} \; dx.$$

$$\frac{d}{dx} \Big\downarrow \quad \boxed{\begin{array}{ll} u = \sin^{-1} x, & v = x \\ u' = \dfrac{1}{\sqrt{1 - x^2}}, & v' = 1 \end{array}} \quad \Big\uparrow \int$$

$$I = [u\,v]_a^b - \int_a^b u'\,v \; dx$$

$$\therefore \quad I = [x \sin^{-1} x]_0^1 - \int_0^1 \frac{x}{\sqrt{1 - x^2}} \; dx$$

The substitution $w = 1 - x^2$ is used to evaluate the integral on the RHS. This gives, remembering to change the limits of integration to w - values,

$$I = [x \sin^{-1} x]_0^1 + (1/2) \int_1^0 w^{-1/2} \; dw$$

$$\Rightarrow \quad I = [x \sin^{-1} x]_0^1 + [w^{1/2}]_1^0$$

$$\Rightarrow \quad I = (1 \,.\, \sin^{-1} 1 - 0 \,.\, \sin^{-1} 0) + (0 - 1)$$

$$\Rightarrow \quad \boxed{I = \pi/2 - 1}$$

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AH Maths - MiA (2nd Edn.)

- pg. 116 Ex. 7.8 Q 1 a, b, d, k, 2 d, g, 4 a, b, 5 a, d, 6 a, b, d.
- pg. 118 Ex. 7.9 Q 1 a, e, 2 a, j.

Ex. 7.8

1 Integrate each function using integration by parts.

a $x \sin x$ **b** xe^x **d** $3x \sec^2 x$ **k** $\sqrt{x} \ln x$

2 Find these integrals, using integration by parts.

d $\int (1 - 4x) \sin (2x - 3)\, dx$

g $\int (2x + 1)e^{4-x}\, dx$

4 Integrate by parts

a $x \tan^{-1} x$ **b** $x \sin^{-1} x$

5 Integrate by parts

a $x^2 \sin x$ **d** $(x + 1)^2 e^x$

6 Evaluate these definite integrals.

a $\int_0^{\pi} x \sin x\, dx$ **b** $\int_0^{e} xe^{3x}\, dx$ **d** $\int_0^{\frac{\pi}{2}} x^2 \sin 2x\, dx$

Ex. 7.9

1

a $\int \ln 3x \, dx$

e $\int \tan^{-1} x \, dx$

2

a $\int e^x \sin x \, dx$

j $\int \frac{\cos 2x}{e^x} \, dx$

Answers to AH Maths (MiA), pg. 116, Ex. 7.8

1 a $-x\cos x + \sin x + c$ **b** $xe^x - e^x + c$

d $3x\tan x + 3\ln|\cos x| + c$

k $\frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + c$

2 d $-\frac{1}{2}(1-4x)\cos(2x-3) - \sin(2x-3) + c$

g $-(2x+1)e^{4-x} - 2e^{4-x} + c$

4 a $\frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + c$

b $\frac{1}{4}(2x^2-1)\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2} + c$

5 a $(2-x^2)\cos x + 2x\sin x + c$

d $e^x(x^2+1) + c$

6 a π

b $\frac{1}{3}e^{3e+1} - \frac{1}{9}e^{3e} + \frac{1}{9}$

d $\frac{\pi^2}{8} - \frac{1}{2}$

Answers to AH Maths (MiA), pg. 118, Ex. 7.9

1 a $x \ln 3x - x + c$

e $x \tan^{-1} x - \frac{1}{2} \ln (1 + x^2) + c$

2 a $\frac{e^x}{2}(\sin x - \cos x) + c$

j $\frac{e^{-x}}{5}(2 \sin 2x - \cos 2x) + c$