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Differentiation and Properties of Functions - Lesson 4

Graph Sketching

LI

- Sketch graphs of polynomials, especially cubics and quartics.

SC

- SPs.
- Intersections with axes.
- Behaviour as $x \longrightarrow \pm\infty$.

Jan 10-12:11

To sketch the graph of a polynomial (indicating stationary points), find :

- x - intercepts (put $y = 0$ then synthetic division to get roots).
- y - intercept (put $x = 0$).
- SPs and their nature.
- Behaviour as $x \longrightarrow \pm\infty$.

Example 1

Sketch the graph of the function
$f(x) = x^3 + 2x^2 + x$, annotating it fully.

'Annotating it fully' means indicating intersections with axes and indicating SPs.

$$f(x) = x^3 + 2x^2 + x$$
$$\Rightarrow \quad f(x) = x(x^2 + 2x + 1)$$
$$\Rightarrow \quad f(x) = x(x + 1)^2$$

- x - intercepts :

$$f(x) = 0$$
$$\Rightarrow \quad x(x + 1)^2 = 0$$
$$\Rightarrow \quad \underline{x = 0, x = -1}$$
$$\therefore \quad \boxed{(0, 0), (0, -1)}$$

- y - intercept :

$$f(x) = x^3 + 2x^2 + x$$
$$\therefore \quad f(0) = (0)^3 + 2(0)^2 + 0$$
$$\Rightarrow \quad \underline{f(0) = 0}$$
$$\therefore \quad \boxed{(0, 0)}$$

- SPs :

$$f'(x) = 0$$
$$\therefore \quad 3x^2 + 4x + 1 = 0$$
$$\Rightarrow \quad (3x + 1)(x + 1) = 0$$
$$\Rightarrow \quad \underline{x = -\frac{1}{3}, x = -1}$$

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$\underline{x = -1:}$

$$f(x) = x^3 + 2x^2 + x$$

$$\therefore \quad f(-1) = (-1)^3 + 2(-1)^2 + (-1)$$

$$\Rightarrow \quad f(-1) = -1 + 2 - 1$$

$$\Rightarrow \quad \underline{f(-1) = 0}$$

$$\therefore \quad \boxed{(-1, 0)}$$

$\underline{x = -\frac{1}{3}:}$

$$f(x) = x^3 + 2x^2 + x$$

$$\therefore \quad f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + 2\left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)$$

$$\Rightarrow \quad f\left(-\frac{1}{3}\right) = -\frac{1}{27} + \frac{2}{9} - \frac{1}{3}$$

$$\Rightarrow \quad \underline{f\left(-\frac{1}{3}\right) = -\frac{4}{27}}$$

$$\therefore \quad \boxed{\left(-\frac{1}{3}, -\frac{4}{27}\right)}$$

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x	$\xrightarrow{-2}$	-1	$\xrightarrow{-\frac{1}{2}}$	$-\frac{1}{3}$	$\xrightarrow{0}$
$f'(x)$	$+$	0	$-$	0	$+$
Slope	/	—	\	—	/

$$f'(x) = (3x + 1)(x + 1)$$

$$\therefore \quad f'(-2) = (-5)(-1)$$

$$\Rightarrow \quad \underline{f'(-2) = 5 > 0}$$

$$f'(x) = (3x + 1)(x + 1)$$

$$\therefore \quad f'\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad \underline{f'\left(-\frac{1}{2}\right) = -\frac{1}{4} < 0}$$

$$f'(x) = (3x + 1)(x + 1)$$

$$\therefore \quad f'(0) = (1)(1)$$

$$\Rightarrow \quad \underline{f'(0) = 1 > 0}$$

$(-1, 0)$ is a local max. and

$\left(-\frac{1}{3}, -\frac{4}{27}\right)$ is a local min.

May 30-10:06

- Behaviour as $x \longrightarrow \pm\infty$:

For large values of x (positive or negative), the x^3 term is dominant, so that,

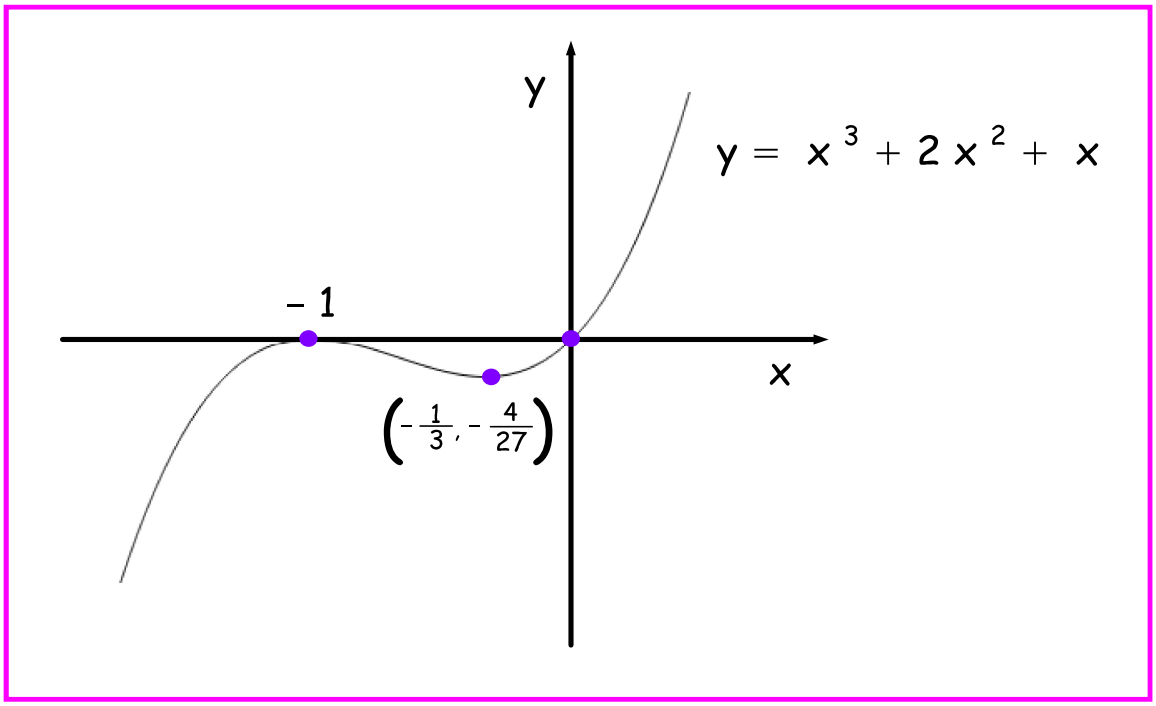
$$y = f(x) \approx x^3$$

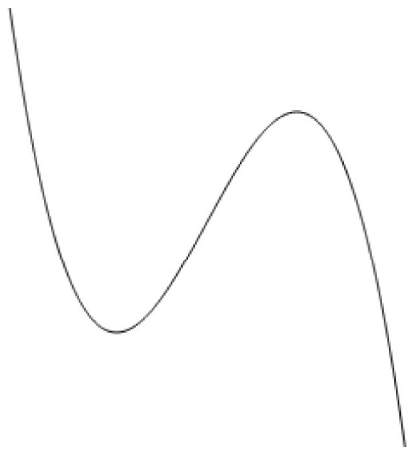
As $x \longrightarrow +\infty$ (i. e. for large and positive x),
$f(x) \longrightarrow +\infty$ (i. e. y is large and positive).

positive x (positive)^3 = positive

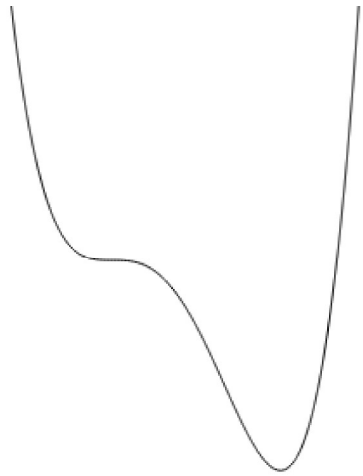
As $x \longrightarrow -\infty$ (i. e. for large and negative x),
$f(x) \longrightarrow -\infty$ (i. e. y is large and negative).

positive x (negative)^3 = negative





Example 2



Example 3

Example 2

Sketch the graph of the function
$f(x) = -x^3 - 3x^2 + 9x + 2.$

	x^3	x^2	x^1	x^0
2	-1	-3	9	2
		-2	-10	-2
	-1	-5	-1	$\boxed{0}$

$$\therefore \quad f(x) = (x-2)(-x^2 - 5x - 1)$$

$$\Rightarrow \quad f(x) = -(x-2)(x^2 + 5x + 1)$$

- x - intercepts :

$$f(x) = 0$$

$$\Rightarrow \quad -(x-2)(x^2 + 5x + 1) = 0$$

$$\Rightarrow \quad \underline{x = 2, x = \frac{-5 \pm \sqrt{21}}{2}}$$

$$\therefore \quad \boxed{(2, 0), \left(\frac{-5 \pm \sqrt{21}}{2}, 0\right)}$$

- y - intercept :

$$f(x) = -x^3 - 3x^2 + 9x + 2$$

$$\therefore \quad f(0) = -(0)^3 - 3(0)^2 + 9(0) + 2$$

$$\Rightarrow \quad \underline{f(0) = 2}$$

$$\therefore \quad \boxed{(0, 2)}$$

- SPs :

$$f'(x) = 0$$

$$\therefore \quad -3x^2 - 6x + 9 = 0$$

$$\Rightarrow \quad -3(x^2 + 2x - 3) = 0$$

$$\Rightarrow \quad -3(x+3)(x-1) = 0$$

$$\Rightarrow \quad \underline{x = -3, x = 1}$$

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$\underline{x = -3:}$

$$f(x) = -x^3 - 3x^2 + 9x + 2$$

$$\therefore \quad f(-3) = -(-3)^3 - 3(-3)^2 + 9(-3) + 2$$

$$\Rightarrow \quad f(-3) = 27 - 27 - 27 + 2$$

$$\Rightarrow \quad \underline{f(-3) = -25}$$

$$\therefore \quad \boxed{(-3, -25)}$$

$\underline{x = 1:}$

$$f(x) = -x^3 - 3x^2 + 9x + 2$$

$$\therefore \quad f(1) = -(1)^3 - 3(1)^2 + 9(1) + 2$$

$$\Rightarrow \quad f(1) = -1 - 3 + 9 + 2$$

$$\Rightarrow \quad \underline{f(1) = 7}$$

$$\therefore \quad \boxed{(1, 7)}$$

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x	$\xrightarrow{-4}$	-3	$\xrightarrow{0}$	1	$\xrightarrow{2}$
$f'(x)$	$-$	0	$+$	0	$-$
Slope	$\diagdown$	—	$\diagup$	—	$\diagdown$

$$f'(x) = -3(x+3)(x-1)$$

$$\therefore \quad f'(-4) = -3(-1)(-5)$$

$$\Rightarrow \quad \underline{f'(-4) = -15 < 0}$$

$$f'(x) = -3(x+3)(x-1)$$

$$\therefore \quad f'(0) = -3(3)(-1)$$

$$\Rightarrow \quad \underline{f'(0) = 9 > 0}$$

$$f'(x) = -3(x+3)(x-1)$$

$$\therefore \quad f'(2) = -3(5)(1)$$

$$\Rightarrow \quad \underline{f'(2) = -15 < 0}$$

$(-3, 25)$ is a local min. and
$(1, 7)$ is a local min.

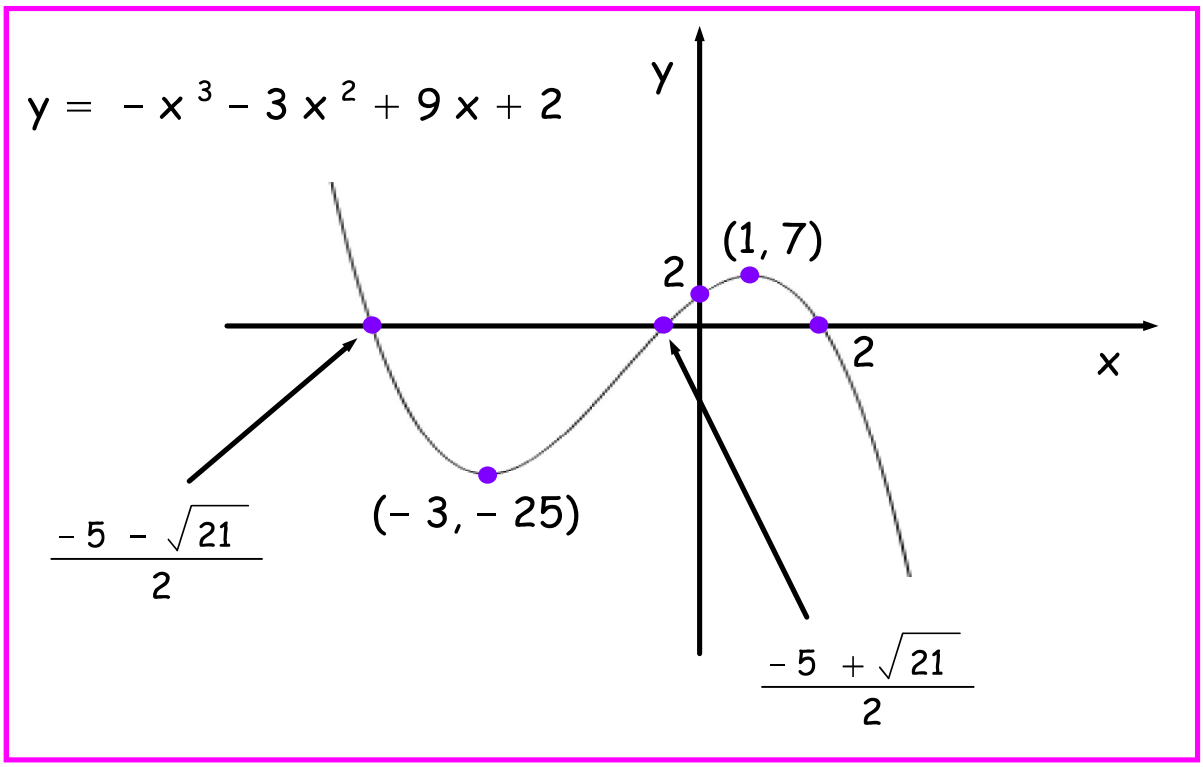
- Behaviour as $x \longrightarrow \pm\infty$:

For large values of x (positive or negative), the x^3 term is dominant, so that,

$$y = f(x) \approx -x^3$$

As $x \longrightarrow +\infty$ (i. e. for large and positive x), $f(x) \longrightarrow -\infty$ (i. e. y is large and negative).

As $x \longrightarrow -\infty$ (i. e. for large and negative x), $f(x) \longrightarrow +\infty$ (i. e. y is large and positive).



Example 3

Sketch the graph of the function
$f(x) = 2x^4 - 8x^3$.

$$f(x) = 2x^4 - 8x^3$$
$$\Rightarrow \quad f(x) = 2x^3(x - 4)$$

- x - intercepts :

$$f(x) = 0$$
$$\Rightarrow \quad 2x^3(x - 4) = 0$$
$$\Rightarrow \quad \underline{x = 0, x = 4}$$
$$\therefore \quad \boxed{(0, 0), (0, 4)}$$

- y - intercept :

$$f(x) = 2x^4 - 8x^3$$
$$\therefore \quad f(0) = 2(0)^4 - 8(0)^3$$
$$\Rightarrow \quad \underline{f(0) = 0}$$
$$\therefore \quad \boxed{(0, 0)}$$

- SPs :

$$f'(x) = 0$$
$$\therefore \quad 8x^3 - 24x^2 = 0$$
$$\Rightarrow \quad 8x^2(x - 3) = 0$$
$$\Rightarrow \quad \underline{x = 0, x = 3}$$

$\underline{x = 0:}$

$$f(x) = 2x^4 - 8x^3$$
$$\therefore \quad f(0) = 2(0)^4 - 8(0)^3$$
$$\Rightarrow \quad \underline{f(0) = 0}$$
$$\therefore \quad \boxed{(0, 0)}$$

$\underline{x = 3:}$

$$f(x) = 2x^4 - 8x^3$$
$$\therefore \quad f(3) = 2(3)^4 - 8(3)^3$$
$$\Rightarrow \quad \underline{f(3) = -54}$$
$$\therefore \quad \boxed{(3, -54)}$$

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x	$\xrightarrow{-1}$	0	$\xrightarrow{1}$	3	$\xrightarrow{4}$
f ' (x)	−	0	−	0	+
Slope	╲	—	╲	—	╱

$$f'(x) = 8x^2(x - 3)$$

$$\therefore \quad f'(-1) = 8(-4)$$

$$\Rightarrow \quad \underline{f'(-1) = -32 < 0}$$

$$f'(x) = 8x^2(x - 3)$$

$$\therefore \quad f'(1) = 8(-2)$$

$$\Rightarrow \quad \underline{f'(1) = -16 < 0}$$

$$f'(x) = 8x^2(x - 3)$$

$$\therefore \quad f'(4) = 128(1)$$

$$\Rightarrow \quad \underline{f'(4) = 128 > 0}$$

$\boxed{(0, 0) \text{ is a P of I and } (3, -54) \text{ is a local min.}}$

May 30-10:06

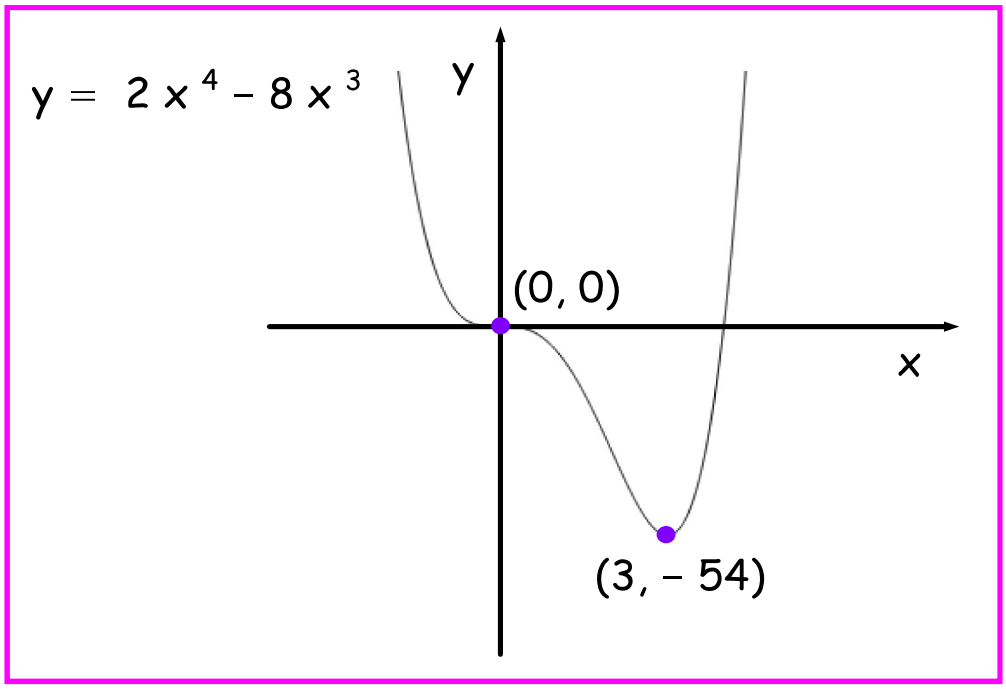
- Behaviour as $x \longrightarrow \pm\infty$:

For large values of x (positive or negative), the x^4 term is dominant, so that,

$$y = f(x) \approx x^4$$

As $x \longrightarrow +\infty$ (i. e. for large and positive x), $f(x) \longrightarrow +\infty$ (i. e. y is large and negative).

As $x \longrightarrow -\infty$ (i. e. for large and negative x), $f(x) \longrightarrow +\infty$ (i. e. y is large and positive).



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