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Unit 2 : Arithmetic and Algebra of Complex Numbers - Lesson 3

Solving Complex Polynomials

LI

- Solve complex cubics and quartics with real coefficients.

SC

- Long division.
- Factor Theorem.

Jan 10-12:11

The Fundamental Theorem of Algebra states that every non-constant polynomial with complex coefficients has at least one complex root

If z is a non-real root of a polynomial P with all coefficients real, then the complex conjugate of z (i.e. $\bar{z}$) is also a root of P

A polynomial of degree n has n complex roots

A polynomial of degree n with all coefficients real can be written as a product of real linear and real irreducible quadratic factors

Possible Roots for a Cubic

- 3 real roots.
- 1 real root, 2 non-real roots (conjugates).

Possible Roots for a Quartic

- 4 real roots.
- 2 real roots, 2 non-real roots (conjugates).
- 2 pairs of non-real roots (conjugates).

Example 1

If $z = 5$ is a root of $z^3 - 11z^2 + 43z - 65$, find the remaining roots.

Let $P(z) = z^3 - 11z^2 + 43z - 65$. As $P(5) = 0$, $z - 5$ is a factor of P.

$$
\begin{array}{r|l}
 & z^2 - 6z + 13 \\
\hline
z - 5 & z^3 - 11z^2 + 43z - 65 \\
 & z^3 - 5z^2 \\
\hline
 & -6z^2 + 43z - 65 \\
 & -6z^2 + 30z \\
\hline
 & 13z - 65 \\
 & 13z - 65 \\
\hline
 & 0
\end{array}
$$

As remainder $= 0$, $z^2 - 6z + 13$ is also a factor of P; so,

$$P(z) = (z - 5)(z^2 - 6z + 13)$$

Roots :

$$P(z) = 0 \Rightarrow (z - 5)(z^2 - 6z + 13) = 0$$

$$\Rightarrow z - 5 = 0,\ z^2 - 6z + 13 = 0$$

Use Quadratic Formula

$$z = 5 \qquad z = 3 + 2i,\ 3 - 2i$$

$\therefore$ $\boxed{\text{Remaining roots : } 3 + 2i \text{ and } 3 - 2i}$

Example 2

Verify that $z = 6 - i$ is a root of the polynomial
$P(z) = z^4 - 10z^3 + 10z^2 + 110z - 111$.

Hence obtain all the roots and factorise P into a product of real linear and real irreducible quadratic factors.

$$(6 - i)^2 = \underline{35 - 12i}$$

$$\begin{aligned}(6 - i)^3 &= (6 - i)^2 (6 - i) \\ &= (35 - 12i)(6 - i) \\ &= \underline{198 - 107i}\end{aligned}$$

$$\begin{aligned}(6 - i)^4 &= (6 - i)^2 (6 - i)^2 \\ &= (35 - 12i)^2 \\ &= \underline{1\,081 - 840i}\end{aligned}$$

$$\begin{aligned}P(6 - i) &= (6 - i)^4 - 10(6 - i)^3 + 10(6 - i)^2 \\ &\quad + 110(6 - i) - 111\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad P(6 - i) &= 1\,081 - 840i - 10(198 - 107i) \\ &\quad + 10(35 - 12i) + 110(6 - i) - 111\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad P(6 - i) &= (1\,081 - 1\,980 + 350 + 660 - 111) \\ &\quad + (-840 + 1\,070 - 120 - 110)i\end{aligned}$$

$$\Rightarrow \quad P(6 - i) = 0 + 0i$$

$$\Rightarrow \quad \underline{P(6 - i) = 0}$$

$$\boxed{\text{As } P(6 - i) = 0,\ 6 - i \text{ is a root of } P}$$

Oct 2-23:10

As $6 - i$ is a root of P and P has all coefficients real, $6 + i$ is also a root of P. So,

$$(z - (6 - i))(z - (6 + i)) \text{ is a factor of } P.$$

$$\begin{aligned}
&(z - (6 - i))(z - (6 + i)) \\
&= ((z - 6) + i))((z - 6) - i)) \\
&= (z - 6)^2 + 1 \\
&= \underline{z^2 - 12z + 37}
\end{aligned}$$

$$\begin{array}{r|l}
 & z^2 + 2z - 3 \\
\hline
z^2 - 12z + 37 & z^4 - 10z^3 + 10z^2 + 110z - 111 \\
 & z^4 - 12z^3 + 37z^2 \\
\hline
 & 2z^3 - 27z^2 + 110z - 111 \\
 & 2z^3 - 24z^2 + 74z \\
\hline
 & -3z^2 + 36z - 111 \\
 & -3z^2 + 36z - 111 \\
\hline
 & \underline{0}
\end{array}$$

As remainder $= 0$, $z^2 + 2z - 3$ is also a factor of P; so,

$$P(z) = (z^2 - 12z + 37)(z^2 + 2z - 3)$$

<u>Roots :</u>

$$P(z) = 0 \Rightarrow (z^2 - 12z + 37)(z^2 + 2z - 3) = 0$$

$$z^2 - 12z + 37 = 0 \qquad z^2 + 2z - 3 = 0$$

$$(z + 3)(z - 1) = 0$$

$$\underline{z = 6 \pm i} \qquad \underline{z = -3,\ z = 1}$$

$$\boxed{\text{All roots : } 6 \pm i, -3, 1}$$

$$\boxed{P(z) = (z^2 - 12z + 37)(z + 3)(z - 1)}$$

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Example 3

Given that $1 + i$ is a root of the polynomial P defined by $P(z) = z^4 - 8z^3 + 39z^2 - 62z + 50$, find all the roots of P.

Hence find the real factors of P.

As $1 + i$ is a root of P and P has all coefficients real, $1 - i$ is also a root of P. So,

$(z - (1 + i))(z - (1 - i))$ is a factor of P.

$$(z - (1 + i))(z - (1 - i))$$
$$= ((z - 1) - i))((z - 1) + i))$$
$$= (z - 1)^2 + 1$$
$$= \underline{z^2 - 2z + 2}$$

$$\begin{array}{r|l}
 & z^2 - 6z + 25 \\
\hline
z^2 - 2z + 2 & z^4 - 8z^3 + 39z^2 - 62z + 50 \\
 & z^4 - 2z^3 + 2z^2 \\
\hline
 & -6z^3 + 37z^2 - 62z + 50 \\
 & -6z^3 + 12z^2 - 12z \\
\hline
 & 25z^2 - 50z + 50 \\
 & 25z^2 - 50z + 50 \\
\hline
 & 0
\end{array}$$

As remainder $= 0$, $z^2 - 6z + 25$ is also a factor of P; so,

$$P(z) = (z^2 - 2z + 2)(z^2 - 6z + 25)$$

Roots :

$$P(z) = 0 \Rightarrow (z^2 - 2z + 2)(z^2 - 6z + 25) = 0$$

$$z^2 - 2z + 2 = 0 \qquad z^2 - 6z + 25 = 0$$

Use Quadratic Formula

$$\underline{z = 1 \pm i} \qquad \underline{z = 3 \pm 4i}$$

$$\boxed{\text{All roots : } 1 \pm i, 3 \pm 4i}$$

$$P(z) = (z^2 - 2z + 2)(z^2 - 6z + 25)$$

$$\boxed{\text{Real factors of } P : z^2 - 2z + 2,\ z^2 - 6z + 25}$$

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AH Maths - MiA (2nd Edn.)

- pg. 224-5 Ex. 12.8 Q 2 - 7.

Feb 10-15:47

Ex. 12.8

2 For each cubic equation, a real root has been identified.
Find the remaining two complex roots.

a $z^3 + 2z^2 + z + 2 = 0;$ $z = -2$

b $z^3 - z^2 - z - 15 = 0;$ $z = 3$

c $z^3 + 6z^2 + 37z + 58 = 0;$ $z = -2$

d $z^3 - 11z - 20 = 0;$ $z = 4$

e $4z^3 + 8z^2 + 14z + 10 = 0;$ $z = -1$

f $2z^3 + 11z^2 + 20z - 13 = 0;$ $z = \frac{1}{2}$

3 For each cubic equation, a factor has been given.
Find the roots of the equation.

a $z^3 + z^2 - 7z - 15 = 0;$ $z - 3$

b $z^3 - 6z^2 + 13z - 20 = 0;$ $z - 4$

c $2z^3 + 5z^2 + 8z + 20 = 0;$ $z - 2i$

d $2z^3 + 7z^2 + 26z + 30 = 0;$ $z + 1 + 3i$

e $4z^3 - 4z^2 + 9z - 9 = 0;$ $2z - 3i$

f $2z^3 + 13z^2 + 46z + 65 = 0;$ $z + 2 - 3i$

4 Given $f(z) = z^4 - z^3 - 2z^2 + 6z - 4$ and given that $z = 1 + i$ is a root of the equation $f(z) = 0$ find the real factors of the polynomial.

5 $1 + 2i$ is a root of the equation $z^4 - 5z^3 + 13z^2 - 19z + 10$.
Find the real factors of the polynomial.

6 For each of these

i show that the given complex number is a zero of the given polynomial

ii find all the remaining roots.

a $z = 2 + i;$ $f(z) = z^4 - 2z^3 - z^2 + 2z + 10$

b $z = 3 + 2i;$ $f(z) = z^4 - 8z^3 + 30z^2 - 56z + 65$

c $z = 1 + 3i;$ $f(z) = 2z^4 - 3z^3 + 17z^2 + 12z - 10$

7 In each case the given complex number is a zero of the given polynomial.
Find all the roots.

a $z = 4 + i;$ $f(z) = z^4 - 8z^3 + 13z^2 + 32z - 68$

b $z = 2 - 3i;$ $f(z) = 6z^4 - 31z^3 + 108z^2 - 99z + 26$

c $z = 4 + 2i;$ $f(z) = 2z^4 - 25z^3 + 107z^2 - 140z - 100$

Answers to AH Maths (MiA), pg. 224-5, Ex. 12.8

2 **a** $\pm i$ **b** $-1 \pm 2i$ **c** $-2 \pm 5i$

d $-2 \pm i$ **e** $-0{\cdot}5 \pm 1{\cdot}5i$ **f** $-3 \pm 2i$

3 **a** $3, -2 \pm i$ **b** $4, 1 \pm 2i$

c $-2{\cdot}5, \pm 2i$ **d** $-1{\cdot}5, -1 \pm 3i$

e $1, \pm 1{\cdot}5i$ **f** $-2{\cdot}5, -2 \pm 3i$

4 $(z + 2)(z - 1), z^2 - 2z + 2$

5 $(z - 2)(z - 1), z^2 - 2z + 5$

6 **a** $2 - i, -1 \pm i$ **b** $3 - 2i, 1 \pm 2i$

c $1 - 3i, -1, 0{\cdot}5$

7 **a** $4 \pm i, \pm 2$ **b** $2 \pm 3i, \frac{2}{3}, \frac{1}{2}$

c $4 \pm 2i, 5, -0{\cdot}5$