Differential Calculus - Lesson 1

Differentiating Powers of a Variable and Basic Rules for Differentiation

LI

- Know the meaning of Differentiation and Derivative.
- Differentiate functions of the form $f(x) = k x^n$.
- Differentiate a sum or difference of 2 or more functions.

SC

- Differentiation rules.
- Rules of indices.
- Rules of fractions.

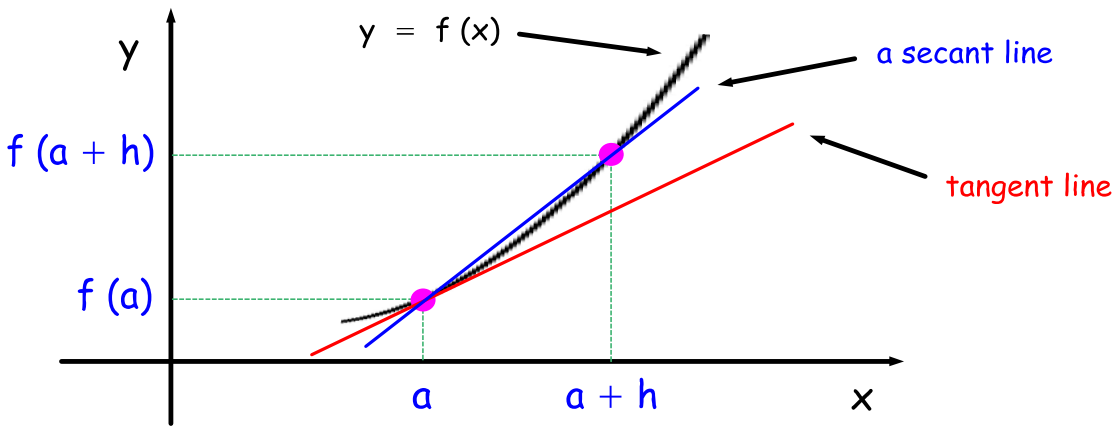
Differential Calculus

Differential Calculus is the study of rates of change; how quickly one quantity changes with respect to another.

The rate of change is called the Derivative.

The process of obtaining the derivative is called Differentiation.

The derivative is obtained as a limit of gradients of secant lines to a curve $y = f(x)$, as shown in the following diagram :



h is a small positive number measured from $x = a$;
a is fixed, but h is allowed to change.

The gradient of the secant line in the above diagram is,

$$m = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}$$

Taking smaller and smaller values for h gives gradients of secant lines that approximate the rate of change of f at $x = a$ more and more accurately.

The rate of change of the function at $x = a$ (aka the derivative of f at $x = a$) tells us how quickly the y - value changes with respect to the x - value at $x = a$ and is defined as :

$$f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

'f dashed of a equals the limit as h goes to zero of the gradient of the secant lines'

<u>Equivalent phrases for the derivative :</u>

Derivative

Derived function

Rate of change

Gradient/slope of tangent

Differential coefficient

In practice, using the above to work out the derivative is normally very difficult, so we use rules that have been worked out for us already by dedicated people.

Different Notations for the Derivative

If $y = f(x)$:

- Lagrange Notation

$y'(x)$	'y dash(ed) of x' or 'y prime of x'
$f'(x)$	
y'	'y dash(ed)' or 'y prime'
f'	

- Leibniz Notation

$\frac{d}{dx} y(x)$	'd by dx of y of x'
$\frac{d}{dx} f(x)$	
$\frac{dy}{dx}$	'dy by dx'
$\frac{df}{dx}$	

Notational warning :
$\frac{dy}{dx}$ is not a fraction

- Euler Notation

$Dy(x)$	'big D of y of x'
$Df(x)$	
Dy	'big D of y'
Df	

Notational warning :
Dy does not mean 'D times y'

- Newton's Notation (normally for time derivatives)

$\dot{y}(t)$ means $\frac{dy}{dt}$	'y dot of t'
$\dot{f}(t)$	
$\dot{y}$	'y dot'
$\dot{f}$	

We mainly use Lagrange (dash) and Leibniz (d by dx) notation.

In this lesson (and the next one), we are only interested in obtaining a formula for the derivative (not working out the derivative at a specific value).

Differentiation Rules

Differentiating a power of x

Leibniz Form :

$$\frac{d}{dx} k x^n = k n x^{n-1}$$

Lagrange Form :

$$(k x^n)' = k n x^{n-1}$$

k is a constant (doesn't involve the variable)

Differentiating a sum or difference of functions

To differentiate a sum/difference of several functions, differentiate each function separately then add/subtract.

Leibniz Form :

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

Lagrange Form :

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

Example 1

If $y = x^7$, find $\frac{dy}{dx}$.

$$y = x^7$$

$$\therefore \quad \boxed{\frac{dy}{dx} = 7x^6}$$

Example 2

Differentiate $\sqrt{x}$.

Let $f(x) = \sqrt{x}$.

$$f(x) = \sqrt{x}$$

$$f(x) = x^{1/2}$$

$$\therefore \quad \boxed{\begin{aligned} f'(x) &= (1/2)\, x^{-1/2} \\ \left(f'(x) = \frac{1}{2\sqrt{x}} \right) \end{aligned}}$$

Example 3

Find the derivative of the function $y = f(x)$ given by $f(x) = 4x^{3/4}$.

$$y = 4x^{3/4}$$

$$\therefore \quad \frac{dy}{dx} = 4(3/4)x^{3/4 - 1}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \frac{dy}{dx} &= 3x^{-1/4} \\ \left(\frac{dy}{dx} = \frac{3}{x^{1/4}}\right) \\ \left(\frac{dy}{dx} = \frac{3}{\sqrt[4]{x}}\right) \end{aligned}}$$

Example 4

If $f(r) = 3r^2 + \dfrac{18}{\sqrt[6]{r}}$, find $\dfrac{df}{dr}$.

$$f(r) = 3r^2 + \frac{18}{\sqrt[6]{r}}$$

$$f(r) = 3r^2 + 18r^{-1/6}$$

$$\therefore \quad \frac{df}{dr} = 3(2)r^{2-1} + 18(-1/6)r^{-1/6-1}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \frac{df}{dr} &= 6r - 3r^{-7/6} \\ \left(\frac{df}{dr} = 6r - \frac{3}{r^{7/6}}\right) \\ \left(\frac{df}{dr} = 6r - \frac{3}{\sqrt[6]{r^7}}\right) \end{aligned}}$$

Example 5

Find $\frac{dp}{dy}$ if $p = -\frac{1}{5}y^{-5} - 12y^{-3/8}$.

$$p = -\frac{1}{5}y^{-5} - 12y^{-3/8}$$

$$\therefore \quad \frac{dp}{dy} = -\frac{1}{5}(-5)y^{-5-1} - 12(-3/8)y^{-3/8-1}$$

$$\Rightarrow \quad \boxed{\begin{gathered} \frac{dp}{dy} = y^{-6} + (9/2)y^{-11/8} \\ \left(\frac{dp}{dy} = \frac{1}{y^6} + \frac{9}{2y^{11/8}}\right) \\ \left(\frac{dp}{dy} = \frac{1}{y^6} + \frac{9}{2\sqrt[8]{y^{11}}}\right) \end{gathered}}$$

Example 6

If $g(n) = \sqrt{n^5} - 11 + 20\,n^{-1/2}$, find the rate of change of g with respect to n.

$$g(n) = \sqrt{n^5} - 11 + 20\,n^{-1/2}$$

$$g(n) = n^{5/2} - 11 + 20\,n^{-1/2}$$

$$\therefore \quad g'(n) = (5/2)\,n^{5/2-1} + 20\,(-1/2)\,n^{-1/2-1}$$

$$\Rightarrow \quad \boxed{\begin{aligned} g'(n) &= (5/2)\,n^{3/2} - 10\,n^{-3/2} \\ \left(g'(n) = \frac{5\,n^{3/2}}{2} - \frac{10}{n^{3/2}} \right) \\ \left(g'(n) = \frac{5\sqrt{n^3}}{2} - \frac{10}{\sqrt{n^3}} \right) \end{aligned}}$$

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Questions

1 Differentiate with respect to x.

a x^8 **b** $-x^4$ **c** x **d** x^{-1} **e** x^{-6} **f** $-x^{-9}$

g 4 **h** $4x^3$ **i** $\frac{3}{4}x^8$ **j** $9x^{-2}$ **k** -5 **l** $-\frac{1}{6}x^{-3}$

m $x^{\frac{3}{2}}$ **n** $x^{\frac{5}{3}}$ **o** $x^{\frac{7}{5}}$ **p** $x^{\frac{1}{2}}$ **q** $x^{\frac{1}{4}}$ **r** $x^{\frac{2}{3}}$

s $x^{-\frac{1}{2}}$ **t** $-x^{-\frac{2}{3}}$ **u** $8x^{\frac{1}{2}}$ **v** $6x^{-\frac{1}{3}}$ **w** $-\frac{5}{3}x^{\frac{3}{2}}$

x $-\frac{1}{4}$ **y** $\frac{3}{7}x^{-\frac{7}{4}}$

2 Differentiate with respect to x.

a $x^2 + 2x - 1$ **b** $-2x^2 - 8x + 3$ **c** $3x - 4$

d $x^3 - 4x^2 + 8x - 10$ **e** $\frac{1}{3}x^3 + 2x^2 - 12x + 4$ **f** $x^4 - 2x^3 + 3x^2 - x - 6$

g $-5x^3 - 2x^{-6}$ **h** $4 - 2x - x^{-6}$ **i** $\frac{1}{2}x^4 + 2x - 3x^{-2}$

j $\frac{3}{10}x^5 - 7 - x^{-1}$

3 Differentiate with respect to x.

a $x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ **b** $12x^{\frac{3}{2}} - x^{\frac{5}{2}}$ **c** $6x^3 - 2x^{\frac{1}{2}}$

d $\frac{2}{5}x^{\frac{5}{4}} - 3x^{-\frac{1}{3}}$ **e** $4x^{-\frac{1}{2}} + \frac{4}{3}x^{-6} - 8x$ **f** $\frac{2}{3}x^{-6} + 8x^{\frac{1}{4}}$

g $8x^{\frac{3}{4}} - \frac{2}{5}x^{-\frac{1}{2}}$ **h** $-4x^{-\frac{1}{4}} + \frac{4}{3}x^{-6} - 8x$ **i** $\frac{4}{5}x^{\frac{1}{4}} - \frac{5}{4}x^{-\frac{1}{5}}$

4 **a** Given $y = x^3 - 4x^2 + x - 8$, find $\frac{dy}{dx}$

b Given $f(x) = x^{\frac{4}{3}}$, find $f'(x)$.

c $y = 6x^{\frac{1}{2}}$. Find the derivative of y with respect to x.

d $f(x) = 2x^2 - 3x - 4$. Find the rate of change of f.

e Differentiate $x^{\frac{3}{4}} - 8x$.

5 Differentiate with respect to the given variable.

a $p^4 - 6p^2 + 9$ **b** $3p^5 - p^{-2}$ **c** $w^3 - 3w + 6$ **d** $5t - 10t^3$

e $8t^{\frac{3}{2}} - 4t^{-\frac{1}{2}}$ **f** $4u^{\frac{2}{3}} - u$ **g** $6t^{-4} - 8t + 7$

Answers

1

a $8x^7$
b $-4x^3$
c 1
d $-\frac{1}{x^2}$
e $-\frac{6}{x^7}$
f $\frac{9}{x^{10}}$
g 0
h $12x^2$
i $6x^7$
j $-\frac{18}{x^3}$
k 0
l $\frac{1}{2x^4}$
m $\frac{3\sqrt{x}}{2}$
n $\frac{5x^{\frac{2}{3}}}{3}$
o $\frac{7x^{\frac{2}{5}}}{5}$
p $\frac{1}{2\sqrt{x}}$
q $\frac{1}{4x^{\frac{3}{4}}}$
r $\frac{2}{3x^{\frac{1}{3}}}$
s $-\frac{1}{2x^{\frac{3}{2}}}$
t $\frac{2}{3x^{\frac{5}{3}}}$
u $\frac{4}{\sqrt{x}}$
v $-\frac{2}{x^{\frac{4}{3}}}$
w $-\frac{5\sqrt{x}}{2}$
x 0
y $-\frac{3}{4x^{\frac{11}{4}}}$

2

a $2 + 2x$
b $-4x - 8$
c 3
d $3x^2 - 8x + 8$
e $x^2 + 4x - 12$
f $4x^3 - 6x^2 + 6x - 1$
g $\frac{12}{x^7} - 15x^2$
h $\frac{6}{x^7} - 2$
i $2x^3 + \frac{6}{x^3} + 2$
j $\frac{3x^4}{2} + \frac{1}{x^2}$

3

a $-\frac{1}{2x^{\frac{3}{2}}} + \frac{1}{2\sqrt{x}}$
b $18\sqrt{x} - \frac{5x^{\frac{3}{2}}}{2}$
c $18x^2 - \frac{1}{\sqrt{x}}$
d $\frac{1}{x^{\frac{4}{3}}} + \frac{x^{\frac{1}{4}}}{2}$
e $-\frac{2}{x^{\frac{3}{2}}} - \frac{8}{x^7} - 8$
f $\frac{2}{x^{\frac{3}{4}}} - \frac{4}{x^7}$
g $\frac{6}{x^{\frac{1}{4}}} + \frac{1}{5x^{\frac{3}{2}}}$
h $\frac{1}{x^{\frac{5}{4}}} - \frac{8}{x^7} - 8$
i $\frac{1}{5x^{\frac{3}{4}}} + \frac{1}{4x^{\frac{6}{5}}}$

4

a $3x^2 - 8x + 1$
b $\frac{4x^{\frac{1}{3}}}{3}$
c $\frac{3}{\sqrt{x}}$
d $4x - 3$
e $\frac{3}{4x^{\frac{1}{4}}} - 8$

5

a $4p^3 - 12p$
b $15p^4 + \frac{2}{p^3}$
c $3w^2 - 3$
d $5 - 30t^2$
e $12\sqrt{t} + \frac{2}{t^{\frac{3}{2}}}$
f $\frac{8}{3u^{\frac{1}{3}}} - 1$
g $-\frac{24}{t^5} - 8$